

A COMBINATORIAL SOLUTION TO MÖGLIN'S PARAMETRIZATION OF ARTHUR PACKETS FOR P-ADIC QUASISPLIT $Sp(N)$ AND $O(N)$

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ABSTRACT. We develop a general procedure to study the combinatorial structure of Arthur packets for p -adic quasisplit $Sp(N)$ and $O(N)$ following the works of Mœglin. This allows us to answer many delicate questions concerning the Arthur packets of these groups, for example the size of the packets.

1. INTRODUCTION

Let F be a p -adic field and G be a quasisplit symplectic or special orthogonal group, i.e., $G = Sp(2n), SO(2n+1)$ and $SO(2n, \eta)$. Here η is a quadratic character associated with a quadratic extension E/F by the local class field theory, and $SO(2n, \eta)$ is the outer form of the split $SO(2n)$ with respect to E/F and an outer automorphism θ_0 induced from the conjugate action of $O(2n)$. We let $\theta_0 = id$ in other cases, and write $\Sigma_0 = \langle \theta_0 \rangle$, $G^{\Sigma_0} = G \rtimes \Sigma_0$. So for $G = SO(2n, \eta)$, $G^{\Sigma_0} \cong O(2n, \eta)$. Note in the case of $SO(8)$, there is another outer form, but we will not consider it in this paper. For simplicity, we will denote $G(F)$ by G , which should not cause any confusion in the context. Let \widehat{G} be the complex dual group of G , and ${}^L G$ be the Langlands dual group of G . Here we can simplify the Langlands dual groups as in the following table:

G	${}^L G$
$Sp(2n)$	$SO(2n+1, \mathbb{C})$
$SO(2n+1)$	$Sp(2n, \mathbb{C})$
$SO(2n, \eta)$	$SO(2n, \mathbb{C}) \rtimes \Gamma_{E/F}$

In the last case, $SO(2n, \mathbb{C}) \rtimes \Gamma_{E/F} \cong O(2n, \mathbb{C})$. So in either of these cases, there is a natural embedding ξ_N of ${}^L G$ into $GL(N, \mathbb{C})$ up to $GL(N, \mathbb{C})$ -conjugacy, where $N = 2n+1$ if $G = Sp(2n)$ or $N = 2n$ otherwise. Let W_F be the Weil group, the local Langlands group can be defined to be

$$L_F := W_F \times SL(2, \mathbb{C}).$$

An Arthur parameter of G is a \widehat{G} -conjugacy class of admissible homomorphisms

$$\underline{\psi} : L_F \times SL(2, \mathbb{C}) \longrightarrow {}^L G,$$

such that $\underline{\psi}|_{W_F}$ is bounded. We denote the set of Arthur parameters of G by $\Psi(G)$. Let $\widehat{\theta}_0$ be the dual automorphism of θ_0 , then Σ_0 acts on $\Psi(G)$ through $\widehat{\theta}_0$, and we denote the corresponding set of Σ_0 -orbits by $\bar{\Psi}(G)$. Let $\Pi(G)$ be the set of equivalence classes of irreducible admissible representations of G , and we denote by $\bar{\Pi}(G)$ the set of Σ_0 -orbits in $\Pi(G)$. For $\psi \in \bar{\Psi}(G)$, Arthur [Art13] shows there exists a finite “multi-set” $\bar{\Pi}_\psi$ of elements in $\bar{\Pi}(G)$, which is related to certain twisted character on $GL(N)$ through the twisted endoscopic character identity (cf. [Xu15a], Section 4). We call $\bar{\Pi}_\psi$ an Arthur packet of G . Mœglin [Mœg11] constructs the elements in $\bar{\Pi}_\psi$, and shows it is in fact **multiplicity free**. As a result, we can also define $\Pi_\psi^{\Sigma_0}$ to be the set of irreducible representations of G^{Σ_0} , whose restriction to G belong to $\bar{\Pi}_\psi$.

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To understand the structure of $\Pi_\psi^{\Sigma_0}$, we need to introduce the set $Jord(\psi)$ of Jordan blocks associated with ψ .

For $\psi \in \bar{\Psi}(G)$, by composing with ξ_N we get an equivalence class of N -dimensional self-dual representation of $L_F \times SL(2, \mathbb{C})$. So we can decompose ψ as follows

$$(1.1) \quad \psi = \bigoplus_{i=1}^r l_i \psi_i = \bigoplus_{i=1}^r l_i (\rho_i \otimes \nu_{a_i} \otimes \nu_{b_i}).$$

Here ρ_i are equivalence classes of irreducible unitary representations of W_F , which can be identified with irreducible unitary supercuspidal representations of $GL(d_{\rho_i})$ under the local Langlands correspondence (cf. [HT01], [Hen00], and [Sch13]). And ν_{a_i} (resp. ν_{b_i}) are the $(a_i - 1)$ -th (resp. $(b_i - 1)$ -th) symmetric power representations of $SL(2, \mathbb{C})$. The irreducible constituent $\rho_i \otimes \nu_{a_i} \otimes \nu_{b_i}$ has dimension $n_i = n_{(\rho_i, a_i, b_i)}$ and multiplicity l_i . We define the multi-set of Jordan blocks for ψ as follows,

$$Jord(\psi) := \{(\rho_i, a_i, b_i) \text{ with multiplicity } l_i : 1 \leq i \leq r\}.$$

Moreover, for any ρ let us define

$$Jord_\rho(\psi) := \{(\rho', a', b') \in Jord(\psi) : \rho' = \rho\}.$$

One can define the parity for self-dual irreducible unitary representations ρ of W_F as in ([Xu15b], Section 3). Then we say (ρ_i, a_i, b_i) is of **orthogonal type** if $a_i + b_i$ is even when ρ_i is of orthogonal type, and $a_i + b_i$ is odd when ρ_i is of symplectic type. Similarly we say (ρ_i, a_i, b_i) is of **symplectic type** if $a_i + b_i$ is odd when ρ_i is of orthogonal type and $a_i + b_i$ is even when ρ_i is of symplectic type. Let ψ_p be the parameter whose Jordan blocks consist of those in $Jord(\psi)$ with the same parity as \hat{G} , and let ψ_{np} be any parameter such that

$$\psi = \psi_{np} \oplus \psi_p \oplus \psi_{np}^\vee,$$

where ψ_{np}^\vee is the dual of ψ_{np} . We also denote by $Jord(\psi)_p$ the set of Jordan blocks in $Jord(\psi_p)$ without multiplicity. Then let us define

$$\widehat{\mathcal{S}}_{\psi>}^{\Sigma_0} = \{\varepsilon(\cdot) \in \mathbb{Z}_2^{Jord(\psi_p)} : \prod_{(\rho, a, b) \in Jord(\psi_p)} \varepsilon(\rho, a, b) = 1\}.$$

and

$$\widehat{\mathcal{S}}_\psi^{\Sigma_0} = \{\varepsilon \in \widehat{\mathcal{S}}_{\psi>}^{\Sigma_0} : \varepsilon(\rho, a, b) = \varepsilon(\rho', a', b') \text{ if } (\rho, a, b) = (\rho', a', b') \text{ in } Jord(\psi)_p\}.$$

If we choose a representative $\underline{\psi} : L_F \times SL(2, \mathbb{C}) \rightarrow {}^L G$, then one can show $\widehat{\mathcal{S}}_\psi^{\Sigma_0}$ is canonically isomorphic to the group of characters of the component group of

$$\text{Cent}(\text{Im } \underline{\psi}, \hat{G} \rtimes \langle \hat{\theta}_0 \rangle) / Z(\hat{G})^{\Gamma_F}.$$

So we will also call elements in $\widehat{\mathcal{S}}_\psi^{\Sigma_0}$ (and also $\widehat{\mathcal{S}}_{\psi>}^{\Sigma_0}$) characters. It follows from Arthur's theory that there is a canonical way to associate any irreducible representation in $\Pi_\psi^{\Sigma_0}$ with an element $\varepsilon \in \widehat{\mathcal{S}}_\psi^{\Sigma_0}$ (cf. [Xu15a], Section 8). Let us denote the direct sum of all irreducible representations associated with $\varepsilon \in \widehat{\mathcal{S}}_\psi^{\Sigma_0}$ by $\pi_W^{\Sigma_0}(\psi, \varepsilon)$, then

$$(1.2) \quad \Pi_\psi^{\Sigma_0} = \bigoplus_{\varepsilon \in \widehat{\mathcal{S}}_\psi^{\Sigma_0}} \pi_W^{\Sigma_0}(\psi, \varepsilon),$$

where we identify $\Pi_\psi^{\Sigma_0}$ with the direct sum of all its elements.

Mœglin's construction of $\Pi_\psi^{\Sigma_0}$ comes with a parametrization by $\widehat{\mathcal{S}}_{\psi>}^{\Sigma_0}$. It also depends on some total order $>_\psi$ on $Jord_\rho(\psi_p)$ for each ρ . To describe the condition on $>_\psi$, we need to write the Jordan blocks differently. For $(\rho, a, b) \in Jord(\psi_p)$, let us write $A = (a + b)/2 - 1$, $B = |a - b|/2$, and set

$\zeta = \zeta_{a,b} = \text{Sign}(a - b)$ if $a \neq b$ and arbitrary otherwise. Then we can denote (ρ, a, b) also by (ρ, A, B, ζ) . We say $>_\psi$ is “admissible” if it satisfies

$$(\mathcal{P}) : \quad \forall (\rho, A, B, \zeta), (\rho, A', B', \zeta') \in \text{Jord}(\psi_p) \text{ with } A > A', B > B' \text{ and } \zeta = \zeta', \\ \text{then } (\rho, A, B, \zeta) >_\psi (\rho, A', B', \zeta').$$

Since the sign ζ is relevant in this condition, Mœglin's parametrization will also depend on the choice of $\zeta_{a,b}$, when $a = b$. First, we have

$$(1.3) \quad \Pi_\psi^{\Sigma_0} = \bigoplus_{\varepsilon \in \widehat{\mathcal{S}}_\psi^{\Sigma_0}} \pi_{M, >_\psi}^{\Sigma_0}(\psi, \varepsilon).$$

The following theorem gives the connection between (1.3) and (1.2).

Theorem 1.1 ([Xu15a], Theorem 8.9). *There exists a character of $\varepsilon_\psi^{M/W} \in \widehat{\mathcal{S}}_\psi^{\Sigma_0}$ such that*

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi, \varepsilon) = \begin{cases} \pi_W^{\Sigma_0}(\psi, \varepsilon \varepsilon_\psi^{M/W}), & \text{if } \varepsilon \varepsilon_\psi^{M/W} \in \widehat{\mathcal{S}}_\psi^{\Sigma_0}, \\ 0, & \text{otherwise.} \end{cases}$$

In [Xu15a], we define

$$\varepsilon_\psi^{M/W} := \varepsilon_\psi^{MW/W} \varepsilon_\psi^{M/MW},$$

for $\varepsilon_\psi^{MW/W}$ and $\varepsilon_\psi^{M/MW}$ in $\widehat{\mathcal{S}}_\psi^{\Sigma_0}$. Here we will recall the definition of these characters.

To define $\varepsilon_\psi^{MW/W}$, we need to first define a set $\mathcal{Z}_{MW/W}(\psi)$ of **unordered pairs** of Jordan blocks from $\text{Jord}(\psi_p)$ as follows. We call a pair $\{(\rho, a, b), (\rho', a', b') \in \text{Jord}(\psi_p)\}$ is contained in $\mathcal{Z}_{MW/W}(\psi)$ if and only if $\rho = \rho'$, and it is in one of the following situations.

(1) Case: a, b are even and a', b' are odd.

$$(a) \text{ If } \zeta_{a,b} = -1 \text{ and } \begin{cases} \zeta_{a',b'} = -1 \Rightarrow (\rho, a, b) >_\psi (\rho, a', b'), a > a' \\ \zeta_{a',b'} = +1 \Rightarrow a > a' \end{cases}$$

$$(b) \text{ If } \zeta_{a,b} = \zeta_{a',b'} = +1 \text{ and } \begin{cases} (\rho, a, b) >_\psi (\rho, a', b') \Rightarrow a' > a, b > b' \\ (\rho, a, b) <_\psi (\rho, a', b') \Rightarrow a > a', b > b' \end{cases}$$

(2) Case : a is odd, b is even and a' is even, b' is odd.

$$(a) \text{ If } \zeta_{a,b} = -1 \text{ and } \begin{cases} \zeta_{a',b'} = -1 \Rightarrow (\rho, a, b) >_\psi (\rho, a', b'), a < a' \\ \zeta_{a',b'} = +1 \text{ and } \begin{cases} (\rho, a, b) >_\psi (\rho, a', b') \Rightarrow a < a' \\ (\rho, a, b) <_\psi (\rho, a', b') \Rightarrow a > a' \end{cases} \end{cases}$$

$$(b) \text{ If } \zeta_{a,b} = \zeta_{a',b'} = +1 \text{ and } \begin{cases} (\rho, a, b) >_\psi (\rho, a', b') \Rightarrow a < a', b > b' \\ (\rho, a, b) <_\psi (\rho, a', b') \Rightarrow a > a', b > b' \end{cases}$$

For $(\rho, a, b) \in \text{Jord}(\psi_p)$, let

$$\mathcal{Z}_{MW/W}(\psi)_{(\rho, a, b)} := \{(\rho', a', b') \in \text{Jord}(\psi_p) : \text{the pair of } (\rho, a, b) \text{ and } (\rho', a', b') \text{ lies in } \mathcal{Z}_{MW/W}(\psi)\}.$$

Then we can define

$$\varepsilon_\psi^{MW/W}(\rho, a, b) := (-1)^{|\mathcal{Z}_{MW/W}(\psi)_{(\rho, a, b)}|}.$$

Next, we define $\varepsilon_\psi^{M/MW}$ according to the following rule. Let $(\rho, a, b) \in \text{Jord}(\psi_p)$.

- (1) If $a + b$ is odd, $\varepsilon_\psi^{M/MW}(\rho, a, b) = 1$.
- (2) If $a + b$ is even, let

$$m = \#\{(\rho, a', b') \in \text{Jord}(\psi) : a', b' \text{ odd}, \zeta_{a',b'} = -1, (\rho, a', b') >_\psi (\rho, a, b)\},$$

and

$$n = \#\{(\rho, a', b') \in \text{Jord}(\psi) : a', b' \text{ odd}, (\rho, a', b') <_\psi (\rho, a, b)\}.$$

Then

$$\varepsilon_{\psi}^{M/MW}(\rho, a, b) = \begin{cases} 1 & \text{if } a, b \text{ even,} \\ (-1)^m & \text{if } a, b \text{ odd, } \zeta_{a,b} = +1, \\ (-1)^{m+n} & \text{if } a, b \text{ odd, } \zeta_{a,b} = -1. \end{cases}$$

Mœglin further parametrizes the irreducible constituents in $\pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \varepsilon)$. To describe that, we need to briefly go through all the stages of Mœglin's construction of $\Pi_{\psi}^{\Sigma_0}$. Let us denote by ψ_d the composition of ψ with

$$\Delta : W_F \times SL(2, \mathbb{C}) \rightarrow W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}),$$

which is the diagonal embedding of $SL(2, \mathbb{C})$ into $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ when restricted to $SL(2, \mathbb{C})$, and is identity on W_F . It is easy to see

$$Jord(\psi_d) = \cup_{(\rho, A, B, \zeta) \in Jord(\psi)} \cup_{C \in [B, A]} \{(\rho, C, C, +1)\}.$$

We call ψ has **discrete diagonal restriction** if $\psi = \psi_p$ and $Jord(\psi_d)$ is multiplicity free. In this case, $Jord_{\rho}(\psi)$ has a natural order $>_{\psi}$, namely

$$(\rho, A, B, \zeta) >_{\psi} (\rho, A', B', \zeta') \text{ if and only if } A > A'.$$

Among the parameters with discrete diagonal restriction, we call ψ is **elementary** if $A = B$ for all $(\rho, A, B, \zeta) \in Jord(\psi)$. For the elementary parameters, Mœglin [Mœg06b] shows $\pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \varepsilon)$ is irreducible.

Suppose ψ has discrete diagonal restriction, Mœglin shows the irreducible constituents of $\pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \varepsilon)$ can be parametrized by pairs of integer-valued functions $(\underline{l}, \underline{\eta})$ over $Jord(\psi)$, such that

$$(1.4) \quad \underline{l}(\rho, A, B, \zeta) \in [0, [(A - B + 1)/2]] \text{ and } \underline{\eta}(\rho, A, B, \zeta) \in \{\pm 1\},$$

and

$$(1.5) \quad \varepsilon(\rho, A, B, \zeta) = \varepsilon_{\underline{l}, \underline{\eta}}(\rho, A, B, \zeta) := \underline{\eta}(\rho, A, B, \zeta)^{A-B+1} (-1)^{[(A-B+1)/2] + \underline{l}(\rho, A, B, \zeta)}.$$

Moreover,

$$\begin{aligned} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) &\hookrightarrow \times_{(\rho, A, B, \zeta) \in Jord(\psi)} \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B + \underline{l}(\rho, A, B, \zeta) - 1) & \cdots & -\zeta(A - \underline{l}(\rho, A, B, \zeta) + 1) \end{pmatrix} \\ &\times \pi_M^{\Sigma_0} \left(\cup_{(\rho, A, B, \zeta) \in Jord(\psi)} \cup_{C \in [B + \underline{l}(\rho, A, B, \zeta), A - \underline{l}(\rho, A, B, \zeta)]} (\rho, C, C, \underline{\eta}(\rho, A, B, \zeta)) (-1)^{C-B-\underline{l}(\rho, A, B, \zeta)} \zeta \right) \end{aligned}$$

as the unique irreducible subrepresentation (see (2.2) and (2.5)). There is an obvious equivalence relation to be made here on pairs $(\underline{l}, \underline{\eta})$, namely

$$(\underline{l}, \underline{\eta}) \sim_{\Sigma_0} (\underline{l}', \underline{\eta}')$$

if and only if $\underline{l} = \underline{l}'$ and $(\underline{\eta}/\underline{\eta}')(\rho, A, B, \zeta) = 1$ unless $\underline{l}(\rho, A, B, \zeta) = (A - B + 1)/2$. Then

$$(1.6) \quad \pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \varepsilon) = \bigoplus_{\{(\underline{l}, \underline{\eta}) : \varepsilon = \varepsilon_{\underline{l}, \underline{\eta}}\} / \sim_{\Sigma_0}} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}).$$

To get to the more general case $\psi = \psi_p$, we need to choose an admissible order $>_{\psi}$ on $Jord(\psi)$. We can index $Jord_{\rho}(\psi)$ such that

$$(\rho, A_i, B_i, \zeta_i) >_{\psi} (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

We say ψ_{\gg} dominates ψ with respect to $>_{\psi}$ if $Jord_{\rho}(\psi_{\gg})$ consists of $(\rho, A_{\gg, i}, B_i + T_{\gg, i}, \zeta_{\gg, i}) := (\rho, A_i + T_i, B_i + T_i, \zeta_i)$ for $T_i \geq 0$, and inherits the same order $>_{\psi}$. We can further choose ψ_{\gg} to have discrete diagonal restriction. After identifying $Jord(\psi)$ with $Jord(\psi_{\gg})$ in the natural way, we can define for any pair of functions $(\underline{l}, \underline{\eta})$ satisfying (1.4) and (1.5),

$$(1.7) \quad \pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) := \circ_{\{\rho : Jord_{\rho}(\psi) \neq \emptyset\}} \circ_{(\rho, A_i, B_i, \zeta_i) \in Jord_{\rho}(\psi)} \text{Jac}_{(\rho, A_i + T_i, B_i + T_i, \zeta_i) \mapsto (\rho, A_i, B_i, \zeta_i)} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}),$$

where i is decreasing in the composition of Jacquet functors (see (2.7)). (This definition is different from that in ([Xu15a], Section 8), for there we take a total order $>_\psi$ on $Jord(\psi)$. But it follows from Lemma 2.2 that only the restriction of $>_\psi$ to $Jord_\rho(\psi)$ for each ρ matters, and the two definition will give the same result.) Then Mœglin shows the following facts (cf. [Xu15a], Proposition 8.5 and Corollary 8.6):

- (1) $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ only depends on the choice of order $>_\psi$, and it is either irreducible or zero.
- (2) If $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \cong \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') \neq 0$, then $(\underline{l}, \underline{\eta}) \sim_{\Sigma_0} (\underline{l}', \underline{\eta}')$.
- (3) The decomposition (1.6) still holds.

Finally, for general $\psi \in \bar{\Psi}(G)$, we have

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi, \varepsilon) = \left(\times_{(\rho, a, b) \in Jord(\psi_{np})} Sp(St(\rho, a), b) \right) \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi_p, \varepsilon)$$

(see (2.3)). Moreover, Mœglin shows

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) := \left(\times_{(\rho, a, b) \in Jord(\psi_{np})} Sp(St(\rho, a), b) \right) \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi_p, \underline{l}, \underline{\eta})$$

is irreducible (cf. [Mœg06a], Theorem 6), when $\pi_{M, >_\psi}^{\Sigma_0}(\psi_p, \underline{l}, \underline{\eta}) \neq 0$.

To summarize, for $\psi \in \bar{\Psi}(G)$ we can refine the decomposition (1.3) as follows

$$(1.8) \quad \Pi_\psi^{\Sigma_0} = \bigoplus_{\{(\underline{l}, \underline{\eta}) : \prod_{(\rho, a, b) \in Jord(\psi_p)} \varepsilon_{\underline{l}, \underline{\eta}}(\rho, a, b) = 1\} / \sim_{\Sigma_0}} \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}).$$

where $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ is either irreducible or zero. So it is natural to ask the following question:

Question 1.2. *When $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$?*

The aim of this paper is to answer this question. To do so, we develop a general procedure (see Section 7), and it will lead to some explicit combinatorial conditions on $(\underline{l}, \underline{\eta})$, which are both necessary and sufficient for $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$. The basic idea in this paper is to consider all admissible orders $>_\psi$, even when one only wants to derive the nonvanishing conditions of $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ with respect to a fixed order. So it is critical to study the change of the parametrization of elements in $\Pi_\psi^{\Sigma_0}$ by $(\underline{l}, \underline{\eta})$, when we vary the order $>_\psi$ (see Section 5). And this also suggests that in general it would be very difficult to compute $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ by using the definition (1.7), for there one has already fixed an order.

Now we want to set up some conventions for this paper. Since only ψ_p is relevant in answering Question 1.2, we will assume $\psi = \psi_p$ in the rest of the paper. We will also require $(\underline{l}, \underline{\eta})$ to always satisfy

$$(1.9) \quad \prod_{(\rho, a, b) \in Jord(\psi)} \varepsilon_{\underline{l}, \underline{\eta}}(\rho, a, b) = 1.$$

So we will not write down this condition later in the paper.

In many arguments of the paper, we need to fix a self-dual irreducible unitary supercuspidal representation ρ of $GL(d_\rho)$. So if ψ_{\gg} is a dominating parameter of ψ , we would like to define

$$\text{Jac}_{X^c} := \circ_{\rho' \neq \rho} \circ_{(\rho', A', B', \zeta') \in Jord_{\rho'}(\psi)} \text{Jac}_{(\rho', A'_{\gg}, B'_{\gg}, \zeta') \mapsto (\rho', A', B', \zeta')}$$

and

$$\mathcal{C}_{X^c} := \times_{\rho' \neq \rho} \times_{(\rho', A', B', \zeta') \in Jord_{\rho'}(\psi)} \begin{pmatrix} \zeta' B'_{\gg} & \cdots & \zeta'(B' + 1) \\ \vdots & & \vdots \\ \zeta' A'_{\gg} & \cdots & \zeta'(A' + 1) \end{pmatrix}.$$

Since we are taking $\rho' \neq \rho$ in Jac_{X^c} (resp. \mathcal{C}_{X^c}), it will “commute” with all kinds of Jacquet functors (resp. induced modules) defined with respect to ρ in our arguments (see Lemma 2.2, Corollary 3.3). Later we will use this property freely without mentioning it.

Finally, for a fixed ρ , we often need to put apart some subset of $Jord_\rho(\psi)$ in different ways. Here we want to quantify the corresponding notions.

- (1) Suppose $(\rho, A, B, \zeta) \in \text{Jord}_\rho(\psi)$ and r is a positive integer, we say (ρ, A, B, ζ) (or $[A, B]$) is in level r “far away”, if

$$B > 2^r \cdot \sum_{(\rho, A', B', \zeta') \in \text{Jord}_\rho(\psi)} (A' - B' + 1),$$

and we write

$$(\rho, A, B, \zeta) \gg_r 0 \text{ or } (\rho, A, B, \zeta) \gg 0 \text{ when } r = 1.$$

- (2) Suppose $(\rho, A, B, \zeta) \in \text{Jord}_\rho(\psi)$ and r is a positive integer, we say (ρ, A, B, ζ) (or $[A, B]$) is in level r “far away” from a subset J of $\text{Jord}_\rho(\psi)$, if

$$B > 2^{r|J|} \cdot \left(\sum_{(\rho, A', B', \zeta') \in J} A' + |J| \sum_{(\rho, A', B', \zeta') \in \text{Jord}_\rho(\psi)} (A' - B' + 1) \right),$$

and we write

$$(\rho, A, B, \zeta) \gg_r J \text{ or } (\rho, A, B, \zeta) \gg J \text{ when } r = 1.$$

- (3) For a subset J of $\text{Jord}_\rho(\psi)$, we denote its complement in $\text{Jord}_\rho(\psi)$ by J^c . We say J is “separated” from J^c , if the following conditions are satisfied.

- (a) For any $(\rho, A, B, \zeta) \in J$ and $(\rho, A', B', \zeta') \in J^c$,

$$\text{either } B' > A \text{ or } B > A'.$$

- (b) For any admissible order $>_J$ on J , there exists a dominating set of Jordan blocks J_{\gg} of J with discrete diagonal restriction, such that for any $(\rho, A, B, \zeta) \in J$ and $(\rho, A', B', \zeta') \in J^c$,

$$\text{if } B' > A \text{ then } B' > A_{\gg}.$$

- (c) There exists an admissible order $>_{J^c}$ on J^c , under which one can find a dominating set of Jordan blocks J_{\gg}^c of J^c with discrete diagonal restriction, such that for any $(\rho, A, B, \zeta) \in J$ and $(\rho, A', B', \zeta') \in J^c$,

$$\text{if } B > A' \text{ then } B > A'_{\gg}.$$

In application, what is important is only the fact that these notions (“far away”, “separated”) can be quantified, but not the specific way that we quantify them. For example, once we can measure what it means for some Jordan blocks to be “far away” from all the others, we can just take them as far as we want in practice.

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2. NOTATION

We will review the notations in [Xu15a], [Xu15b]. For $GL(n)$, let us take B to be the group of upper-triangular matrices and T to be the group of diagonal matrices, then the standard Levi subgroup M can be identified with

$$GL(n_1) \times \cdots \times GL(n_r)$$

for any partition of $n = n_1 + \cdots + n_r$ as follows

$$\begin{pmatrix} GL(n_1) & & \\ & \ddots & \\ & & GL(n_r) \end{pmatrix}$$

$$(g_1, \cdots, g_r) \longrightarrow \text{diag}\{g_1, \cdots, g_r\}.$$

For $\pi = \pi_1 \otimes \cdots \otimes \pi_r$, where π_i is a finite-length admissible representation of $GL(n_i)$ for $1 \leq i \leq r$, we denote the normalized parabolic induction $\text{Ind}_P^G(\pi)$ by

$$\pi_1 \times \cdots \times \pi_r.$$

An irreducible supercuspidal representation of a general linear group can always be written in a unique way as $\rho||^x := \rho \otimes |\det(\cdot)|^x$ for an irreducible unitary supercuspidal representation ρ and a real number x . For a finite length arithmetic progression of real numbers of common length 1 or -1

$$x, \dots, y$$

and an irreducible unitary supercuspidal representation ρ of $GL(d_\rho)$, it is a general fact that

$$\rho||^x \times \dots \times \rho||^y$$

has a unique irreducible subrepresentation, denoted by $\langle \rho; x, \dots, y \rangle$ or $\langle x, \dots, y \rangle$. If $x \geq y$, it is called a Steinberg representation; if $x < y$, it is called a Speh representation. Such sequence of ordered numbers is called a **segment**, and we denote it by $[x, y]$ or $\{x, \dots, y\}$. In particular, when $x = -y > 0$, we can let $a = 2x + 1 \in \mathbb{Z}$ and write

$$St(\rho, a) := \langle \frac{a-1}{2}, \dots, -\frac{a-1}{2} \rangle.$$

We also define a **generalized segment** to be a matrix

$$(2.1) \quad \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}$$

such that each row is a decreasing (resp. increasing) segment and each column is an increasing (resp. decreasing) segment. The normalized induction

$$\times_{i \in [1, m]} \langle \rho; x_{i1}, \dots, x_{in} \rangle$$

has a unique irreducible subrepresentation, and we denote it by $\langle \rho; \{x_{ij}\}_{m \times n} \rangle$. If there is no ambiguity with ρ , we will also write it as $\langle \{x_{ij}\}_{m \times n} \rangle$ or

$$(2.2) \quad \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}$$

Moreover,

$$\langle \rho; \{x_{ij}\}_{m \times n} \rangle \cong \langle \rho; \{x_{ij}\}_{m \times n}^T \rangle$$

where $\{x_{ij}\}_{m \times n}^T$ is the transpose of $\{x_{ij}\}_{m \times n}$. The dual of $\langle \rho; \{x_{ij}\}_{m \times n} \rangle$ is

$$\langle \rho; \{x_{ij}\}_{m \times n} \rangle^\vee \cong \begin{pmatrix} -x_{mn} & \cdots & -x_{m1} \\ \vdots & & \vdots \\ -x_{1n} & \cdots & -x_{11} \end{pmatrix}.$$

Let a, b be positive integers, we define $Sp(St(\rho, a), b)$ to be the unique irreducible subrepresentation of

$$(2.3) \quad St(\rho, a)||^{-(b-1)/2} \times St(\rho, a)||^{-(b-3)/2} \times \dots \times St(\rho, a)||^{(b-1)/2}.$$

Then one can see $Sp(St(\rho, a), b)$ is given by the following generalized segment

$$\begin{bmatrix} (a-b)/2 & \cdots & 1 - (a+b)/2 \\ \vdots & & \vdots \\ (a+b)/2 - 1 & \cdots & -(a-b)/2 \end{bmatrix}.$$

If $G = Sp(2n)$, let us define it with respect to

$$\begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix},$$

where

$$J_n = \begin{pmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{pmatrix}.$$

Let us take B to be subgroup of upper-triangular matrices in G and T to be subgroup of diagonal matrices in G , then the standard Levi subgroup M can be identified with

$$GL(n_1) \times \cdots \times GL(n_r) \times G_-$$

for any partition $n = n_1 + \cdots + n_r + n_-$ and $G_- = Sp(2n_-)$ as follows

$$\begin{pmatrix} GL(n_1) & & & & & 0 \\ & \ddots & & & & \\ & & GL(n_r) & & & \\ & & & G_- & & \\ & & & & GL(n_r) & \\ 0 & & & & & \ddots & \\ & & & & & & GL(n_1) \end{pmatrix}$$

$$(2.4) \quad (g_1, \cdots, g_r, g) \longrightarrow \text{diag}\{g_1, \cdots, g_r, g, {}_t g_r^{-1}, \cdots, {}_t g_1^{-1}\},$$

where ${}_t g_i = J_{n_i} {}^t g_i J_{n_i}^{-1}$ for $1 \leq i \leq r$. Note n_- can be 0, in which case we simply write $Sp(0) = 1$. For $\pi = \pi_1 \otimes \cdots \otimes \pi_r \otimes \sigma$, where π_i is a finite-length admissible representation of $GL(n_i)$ for $1 \leq i \leq r$ and σ is a finite-length admissible representation of G_- , we denote the normalized parabolic induction $\text{Ind}_P^G(\pi)$ by

$$(2.5) \quad \pi_1 \times \cdots \times \pi_r \rtimes \sigma.$$

These notations can be easily extended to special orthogonal groups. If $G = SO(N)$ split, we define it with respect to J_N . When N is odd, the situation is exactly the same as the symplectic case. When $N = 2n$, there are two distinctions. First, the standard Levi subgroups given through the embedding (2.4) do not exhaust all standard Levi subgroups of $SO(2n)$. To get all of them, we need to take the θ_0 -conjugate of M given in (2.4), where

$$\theta_0 = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Note $M^{\theta_0} \neq M$ only when $n_- = 0$ and $n_r > 1$. In this paper, we will only take those Levi subgroups M given in (2.4). Second, if the partition $n = n_1 + \cdots + n_r + n_-$ satisfies $n_r = 1$ and $n_- = 0$, then we can rewrite it as $n = n_1 + \cdots + n_{r-1} + n'_-$ with $n'_- = 1$, and the corresponding Levi subgroup is the same. This is because $GL(1) \cong SO(2)$. If $G = SO(2n, \eta)$, the standard Levi subgroups of $SO(2n, \eta)$ will be the outer form of those θ_0 -stable standard Levi subgroups of $SO(2n)$. In particular, they can be identified with $GL(n_1) \times \cdots \times GL(n_r) \times SO(n_-, \eta)$ and $n_- \neq 0$.

Next we want to define parabolic induction and Jacquet module for the category $\text{Rep}(G^{\Sigma_0})$ of finite-length representations of G^{Σ_0} . Let $P = MN$ be a standard parabolic subgroup of G . If M is θ_0 -stable, we write $M^{\Sigma_0} := M \rtimes \Sigma_0$. Otherwise, we let $M^{\Sigma_0} = M$. Suppose $\sigma^{\Sigma_0} \in \text{Rep}(M^{\Sigma_0})$, $\pi^{\Sigma_0} \in \text{Rep}(G^{\Sigma_0})$.

- (1) If $M^{\theta_0} = M$, we define the normalized parabolic induction $\text{Ind}_{P^{\Sigma_0}}^{G^{\Sigma_0}} \sigma^{\Sigma_0}$ to be the extension of the representation $\text{Ind}_P^G(\sigma^{\Sigma_0}|_M)$ by an induced action of Σ_0 , and we define the normalized Jacquet

module $\text{Jac}_{P\Sigma_0}\pi^{\Sigma_0}$ to be the extension of the representation $\text{Jac}_P(\pi^{\Sigma_0}|_G)$ by an induced action of Σ_0 .

- (2) If $M^{\theta_0} \neq M$, we define the normalized parabolic induction $\text{Ind}_{P\Sigma_0}^{G\Sigma_0}\sigma^{\Sigma_0}$ to be $\text{Ind}_G^{G\Sigma_0}\text{Ind}_P^G(\sigma^{\Sigma_0}|_M)$, and we define the normalized Jacquet module $\text{Jac}_{P\Sigma_0}\pi^{\Sigma_0}$ to be $\text{Jac}_P(\pi^{\Sigma_0}|_G)$.

Let ρ be an irreducible unitary supercuspidal representation of $GL(d_\rho)$, and $M = GL(d_\rho) \times G_-$ be the Levi component of a standard maximal parabolic subgroup P of G . For $\pi^{\Sigma_0} \in \text{Rep}(G^{\Sigma_0})$, we can decompose the semisimplification of the Jacquet module

$$s.s.\text{Jac}_{P\Sigma_0}(\pi) = \bigoplus_i \tau_i \otimes \sigma_i^{\Sigma_0},$$

where $\tau_i \in \text{Rep}(GL(d_\rho))$ and $\sigma_i \in \text{Rep}(G_-^{\Sigma_0})$, both of which are irreducible. We define $\text{Jac}_x\pi^{\Sigma_0}$ for any real number x to be

$$(2.6) \quad \text{Jac}_x(\pi) = \bigoplus_{\tau_i = \rho||^x} \sigma_i^{\Sigma_0}.$$

If we have an ordered sequence of real numbers $\{x_1, \dots, x_s\}$, we can define

$$\text{Jac}_{x_1, \dots, x_s}\pi^{\Sigma_0} = \text{Jac}_{x_s} \circ \dots \circ \text{Jac}_{x_1}\pi^{\Sigma_0}.$$

For a generalized segment X (cf. (2.1)), we define $\text{Jac}_X := \circ_{x \in X} \text{Jac}_x$, where x ranges over X from top to bottom and left to right. Similarly, we can define Jac_x^{op} analogous to Jac_x , but with respect to ρ^\vee and the standard Levi subgroup $GL(n_-) \times GL(d_\rho)$.

For $\psi \in \Psi(G)$, let ψ_{\gg} be a dominating parameter of ψ with respect to certain admissible order $>_\psi$. For $(\rho, A, B, \zeta) \in \text{Jord}(\psi)$, we define

$$(2.7) \quad \text{Jac}_{(\rho, A_{\gg}, B_{\gg}, \zeta) \mapsto (\rho, A, B, \zeta)} := \text{Jac}_{X_{(\rho, A, B, \zeta)}^{\gg}}$$

where

$$X_{(\rho, A, B, \zeta)}^{\gg} = \begin{bmatrix} \zeta B_{\gg} & \dots & \zeta(B+1) \\ \vdots & & \vdots \\ \zeta A_{\gg} & \dots & \zeta(A+1) \end{bmatrix}.$$

The following lemma is very useful when we want to permute the Jacquet functors defined in (2.6).

Lemma 2.1 ([Xu15b], Lemma 5.6). *If $\pi^{\Sigma_0} \in \text{Rep}(G^{\Sigma_0})$ and $|x - y| \neq 1$, then*

$$\text{Jac}_{x,y}\pi^{\Sigma_0} = \text{Jac}_{y,x}\pi^{\Sigma_0}.$$

Lemma 2.2. *Let ρ, ρ' be two distinct unitary irreducible supercuspidal representations of general linear groups, and x, y be any two real numbers. For $\pi^{\Sigma_0} \in \text{Rep}(G^{\Sigma_0})$,*

$$\text{Jac}'_y \circ \text{Jac}_x\pi^{\Sigma_0} = \text{Jac}_x \circ \text{Jac}'_y\pi^{\Sigma_0},$$

where Jac_x (resp. Jac'_y) is defined with respect to ρ (resp. ρ').

Proof. The proof is the same as Lemma 2.1. □

There are some explicit formulas for computing the Jacquet modules in the case of classical groups and general linear groups (cf. [Xu15b], Section 5). Since we will use them quite often, let us recall them here. We will fix a unitary irreducible supercuspidal representation ρ of $GL(d_\rho)$, and take “ $\stackrel{s.s.}{=}$ ” for equality after semisimplification.

For any decreasing segment $\{a, \dots, b\}$ and $\zeta = \pm 1$,

$$\text{Jac}_x \langle \rho'; \zeta a, \dots, \zeta b \rangle = \begin{cases} \langle \rho'; \zeta(a-1), \dots, \zeta b \rangle, & \text{if } x = \zeta a \text{ and } \rho' \cong \rho, \\ 0, & \text{otherwise;} \end{cases}$$

and

$$\text{Jac}_x^{op} < \rho'; \zeta a, \dots, \zeta b > = \begin{cases} < \rho'; \zeta a, \dots, \zeta(b+1) >, & \text{if } x = \zeta b \text{ and } \rho' \cong \rho^\vee, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $\pi_i \in \text{Rep}(GL(n_i))$ for $i = 1$ or 2 , we have

$$\text{Jac}_x(\pi_1 \times \pi_2) \stackrel{s.s.}{=} (\text{Jac}_x \pi_1) \times \pi_2 \oplus \pi_1 \times (\text{Jac}_x \pi_2),$$

and

$$\text{Jac}_x^{op}(\pi_1 \times \pi_2) \stackrel{s.s.}{=} (\text{Jac}_x^{op} \pi_1) \times \pi_2 \oplus \pi_1 \times (\text{Jac}_x^{op} \pi_2).$$

Suppose $\pi^{\Sigma_0} \in \text{Rep}(G)$ and $\tau \in \text{Rep}(GL(d))$, we have

$$\text{Jac}_x(\tau \rtimes \pi^{\Sigma_0}) \stackrel{s.s.}{=} (\text{Jac}_x \tau) \rtimes \pi^{\Sigma_0} \oplus (\text{Jac}_{-x}^{op} \tau) \rtimes \pi^{\Sigma_0} \oplus \tau \rtimes \text{Jac}_x \pi^{\Sigma_0}.$$

Finally, we want to recall the following vanishing result for Jacquet modules of elements in the Arthur packets. This will become very useful when we want to simplify the results of Jacquet modules after applying the above formulas.

Proposition 2.3 ([Xu15a], Proposition 8.3). *Suppose $\psi \in \bar{\Psi}(G)$ and $\pi^{\Sigma_0} \in \Pi_\psi^{\Sigma_0}$. Let ρ be a unitary irreducible supercuspidal representation of $GL(d_\rho)$.*

(1) *For $\zeta \in \{\pm 1\}$ and segment $[x, y]$ with $0 \leq x \leq y$,*

$$\text{Jac}_{\zeta x, \dots, \zeta y} \pi^{\Sigma_0} = 0,$$

unless there exists a sequence of Jordan blocks $\{(\rho, A_i, B_i, \zeta)\}_{i=1}^n \subseteq \text{Jord}_\rho(\psi)$ such that $B_1 = x, A_n > y$, and $B_i \leq B_{i+1} \leq A_i + 1$.

(2) *For $x \in \mathbb{R}$, let $m = \sharp\{(\rho, A, B, \zeta) \in \text{Jord}(\psi) : \zeta B = x\}$. If $n > m$, then*

$$\underbrace{\text{Jac}_x, \dots, x}_n \pi^{\Sigma_0} = 0.$$

3. SOME IRREDUCIBILITY RESULTS

In this section, we want to recall some irreducibility results. We will start with general linear groups. For any two segments $[x, y]$ and $[x', y']$ such that $(x - y)(x' - y') \geq 0$, we say they are **linked** if as sets $[x, y] \not\subseteq [x', y']$, $[x', y'] \not\subseteq [x, y]$, and $[x, y] \cup [x', y']$ can form a segment after imposing the same order. The following theorem is fundamental in determining the reducibility of an induced representation of $GL(n)$.

Theorem 3.1 (Zelevinsky [Zel80]). *For unitary irreducible supercuspidal representations ρ, ρ' of general linear groups, and segments $[x, y], [x', y']$ such that $(x - y)(x' - y') \geq 0$,*

$$< \rho; x, \dots, y > \times < \rho'; x', \dots, y' >$$

is reducible if and only if $\rho \cong \rho'$ and $[x, y], [x', y']$ are linked. In case it is reducible, it consists of the unique irreducible subrepresentations of

$$< \rho; x, \dots, y > \times < \rho; x', \dots, y' > \quad \text{and} \quad < \rho; x', \dots, y' > \times < \rho; x, \dots, y >.$$

To extend this theorem to generalized segments, we have to extend the notion of “link” first. For any two generalized segments $\{x_{ij}\}_{m \times n}$ and $\{y_{ij}\}_{m' \times n'}$ with the same monotone properties for the rows and columns, we say they are **linked** if $[x_{m1}, x_{1n}]$, $[y_{m'1}, y_{1n'}]$ are linked, and the four sides of the rectangle formed by $\{x_{ij}\}_{m \times n}$ do not have inclusive relations with the corresponding four sides of the rectangle formed by $\{y_{ij}\}_{m' \times n'}$ (e.g., $[x_{11}, x_{1n}] \not\subseteq [y_{11}, y_{1n'}]$ and $[x_{11}, x_{1n}] \not\supseteq [y_{11}, y_{1n'}]$, etc). It is easy to check that if $\{x_{ij}\}_{m \times n}$ and $\{y_{ij}\}_{m' \times n'}$ are linked, then $\{x_{ij}\}_{m \times n}^T$ and $\{y_{ij}\}_{m' \times n'}^T$ are also linked. So for generalized segments $\{x_{ij}\}_{m \times n}$ and $\{y_{ij}\}_{m' \times n'}$ with different monotone properties for the rows and columns, we say they are **linked** if $\{x_{ij}\}_{m \times n}^T$ and $\{y_{ij}\}_{m' \times n'}^T$ are linked, or equivalently $\{x_{ij}\}_{m \times n}$ and $\{y_{ij}\}_{m' \times n'}^T$ are linked. One can check this notion of “link” is equivalent to the one in [MW89].

Theorem 3.2 (Mœglin-Waldspurger [MW89]). *For unitary irreducible supercuspidal representations ρ, ρ' of general linear groups, and generalized segments $\{x_{ij}\}_{m \times n}, \{y_{ij}\}_{m' \times n'}$,*

$$< \rho; \{x_{ij}\}_{m \times n} > \times < \rho'; \{y_{ij}\}_{m' \times n'} >$$

is irreducible unless $\rho \cong \rho'$ and $\{x_{ij}\}_{m \times n}, \{y_{ij}\}_{m' \times n'}$ are linked.

We will be mostly using the following corollary of this theorem.

Corollary 3.3. *Let ρ, ρ' be unitary irreducible supercuspidal representations of general linear groups, and $\{x_{ij}\}_{m \times n}, \{y_{ij}\}_{m' \times n'}$ be generalized segments. Suppose $\rho \not\cong \rho'$, or $\{x_{ij}\}_{m \times n}, \{y_{ij}\}_{m' \times n'}$ are not linked, then*

$$< \rho; \{x_{ij}\}_{m \times n} > \times < \rho'; \{y_{ij}\}_{m' \times n'} > \cong < \rho'; \{y_{ij}\}_{m' \times n'} > \times < \rho; \{x_{ij}\}_{m \times n} > .$$

Proof. One just needs to notice there a Weyl group action transform the inducing representation $< \rho; \{x_{ij}\}_{m \times n} > \otimes < \rho'; \{y_{ij}\}_{m' \times n'} >$ to $< \rho'; \{y_{ij}\}_{m' \times n'} > \otimes < \rho; \{x_{ij}\}_{m \times n} >$. Then the corollary follows from the fact that both induced representations are irreducible. \square

Next, let us consider G^{Σ_0} .

Lemma 3.4 ([Mœg11], Lemma 8.2). *Let $\psi \in \bar{\Psi}(G)$ and $\pi^{\Sigma_0} \in \Pi_{\psi}^{\Sigma_0}$. For any self-dual irreducible unitary supercuspidal representation ρ of $GL(d_{\rho})$ and real number x ,*

$$\rho ||^x \rtimes \pi^{\Sigma_0}$$

is irreducible, provided for all $(\rho, A, B, \zeta) \in \text{Jord}_{\rho}(\psi)$, we have either

$$B > |x| \text{ or } |x| > A + 1.$$

We will not give the proof of this lemma here, but we would like to discuss the idea behind the proof. Let τ be an irreducible representation of $GL(d)$, and π^{Σ_0} be an irreducible representation of G^{Σ_0} . To show $\tau \rtimes \pi^{\Sigma_0}$ is irreducible, there is the following criterion.

Lemma 3.5. *Suppose there exists a unique irreducible subrepresentation*

$$\sigma \hookrightarrow \tau \rtimes \pi^{\Sigma_0}$$

such that σ is multiplicity free in $s.s.(\tau \rtimes \pi^{\Sigma_0})$, and

$$\sigma \hookrightarrow \tau^{\vee} \rtimes \pi^{\Sigma_0}.$$

Then $\tau \rtimes \pi^{\Sigma_0}$ is irreducible.

Proof. Since $\sigma \hookrightarrow \tau^{\vee} \rtimes \pi^{\Sigma_0}$, we know $\tau \rtimes \pi^{\Sigma_0}$ has a quotient isomorphic to σ . Then by the fact that $\sigma \hookrightarrow \tau \rtimes \pi^{\Sigma_0}$ and σ is multiplicity free in $s.s.(\tau \rtimes \pi^{\Sigma_0})$, we see σ is a direct summand of $\tau \rtimes \pi^{\Sigma_0}$. This means $\tau \rtimes \pi^{\Sigma_0}$ necessarily has another irreducible subrepresentation. But this contradicts to the uniqueness of σ . \square

By the same idea, we can generalize Lemma 3.4 to the following proposition.

Proposition 3.6. *Let $\psi \in \bar{\Psi}(G)$ and $\pi^{\Sigma_0} \in \Pi_{\psi}^{\Sigma_0}$. For any self-dual irreducible unitary supercuspidal representation ρ of $GL(d_{\rho})$, and*

$$\tau = \begin{pmatrix} \zeta x & \cdots & \zeta x' \\ \vdots & & \vdots \\ \zeta y & \cdots & \zeta y' \end{pmatrix}$$

such that $y \geq x \geq x' > 0$ and $\zeta = \pm 1$, we have $\tau \rtimes \pi^{\Sigma_0}$ is irreducible, provided for all $(\rho, A, B, \zeta) \in \text{Jord}_{\rho}(\psi)$, we have either

$$B > y \text{ or } x' > A + 1.$$

Proof. To apply Lemma 3.5, let us choose an irreducible subrepresentation $\sigma \hookrightarrow \tau \rtimes \pi^{\Sigma_0}$. Let

$$X = \begin{bmatrix} \zeta x & \cdots & \zeta x' \\ \vdots & & \vdots \\ \zeta y & \cdots & \zeta y' \end{bmatrix},$$

then by our assumption, $\text{Jac}_z \pi^{\Sigma_0} = 0$ for any $z \in X$. Also because $y \geq x \geq x' > 0$, we have

$$\text{Jac}_X(\tau \rtimes \pi^{\Sigma_0}) \stackrel{s.s.}{=} (\text{Jac}_X \tau) \rtimes \pi^{\Sigma_0} = \pi^{\Sigma_0}.$$

This means σ is the unique irreducible subrepresentation of $\tau \rtimes \pi^{\Sigma_0}$, and it is multiplicity free in $s.s.(\tau \rtimes \pi^{\Sigma_0})$. Then it suffices for us to show $\sigma \hookrightarrow \tau^\vee \rtimes \pi^{\Sigma_0}$. By Lemma 3.4, we have

$$\begin{aligned} \sigma &\hookrightarrow \begin{pmatrix} \zeta x & \cdots & \zeta x' \\ \vdots & & \vdots \\ \zeta(y-1) & \cdots & \zeta(y'-1) \end{pmatrix} \times \rho^{||^{\zeta y} \times \cdots \times \rho^{||^{\zeta y'}} \rtimes \pi^{\Sigma_0} \\ &\cong \begin{pmatrix} \zeta x & \cdots & \zeta x' \\ \vdots & & \vdots \\ \zeta(y-1) & \cdots & \zeta(y'-1) \end{pmatrix} \times \rho^{||^{\zeta y} \times \cdots \times \rho^{||^{\zeta(y'+1)}} \times \rho^{||^{-\zeta y'}} \rtimes \pi^{\Sigma_0} \\ &\cong \rho^{||^{-\zeta y'}} \times \begin{pmatrix} \zeta x & \cdots & \zeta x' \\ \vdots & & \vdots \\ \zeta(y-1) & \cdots & \zeta(y'-1) \end{pmatrix} \times \rho^{||^{\zeta y} \times \cdots \times \rho^{||^{\zeta(y'+1)}} \rtimes \pi^{\Sigma_0} \\ &\dots \dots \\ &\cong \rho^{||^{-\zeta y'}} \times \cdots \times \rho^{||^{-\zeta y}} \times \begin{pmatrix} \zeta x & \cdots & \zeta x' \\ \vdots & & \vdots \\ \zeta(y-1) & \cdots & \zeta(y'-1) \end{pmatrix} \rtimes \pi^{\Sigma_0} \end{aligned}$$

By induction on $y - x$, we can assume

$$\sigma' := \begin{pmatrix} \zeta x & \cdots & \zeta x' \\ \vdots & & \vdots \\ \zeta(y-1) & \cdots & \zeta(y'-1) \end{pmatrix} \rtimes \pi^{\Sigma_0} \cong \begin{pmatrix} -\zeta(y'-1) & \cdots & -\zeta(y-1) \\ \vdots & & \vdots \\ -\zeta x' & \cdots & -\zeta x \end{pmatrix} \rtimes \pi^{\Sigma_0}$$

is irreducible. Since $\text{Jac}_z \sigma' = 0$ for $z \in [-\zeta y', -\zeta y]$, then

$$\text{Jac}_{-\zeta y', \dots, -\zeta y}(\rho^{||^{-\zeta y'}} \times \cdots \times \rho^{||^{-\zeta y}} \times \sigma') = \sigma'.$$

Therefore,

$$\sigma \hookrightarrow \rho^{||^{-\zeta y'}} \times \cdots \times \rho^{||^{-\zeta y}} \times \sigma'$$

as the unique irreducible subrepresentation. It follows

$$\sigma \hookrightarrow \langle -\zeta y', \dots, -\zeta y \rangle \rtimes \sigma' \cong \langle -\zeta y', \dots, -\zeta y \rangle \rtimes \begin{pmatrix} -\zeta(y'-1) & \cdots & -\zeta(y-1) \\ \vdots & & \vdots \\ -\zeta x' & \cdots & -\zeta x \end{pmatrix} \rtimes \pi^{\Sigma_0}$$

as the unique irreducible subrepresentation. Hence

$$\sigma \hookrightarrow \begin{pmatrix} -\zeta y' & \cdots & -\zeta y \\ \vdots & & \vdots \\ -\zeta x' & \cdots & -\zeta x \end{pmatrix} \rtimes \pi^{\Sigma_0}$$

This finishes the proof. □

We will use the following consequence of this proposition.

Corollary 3.7. *Under the same assumption as Proposition 3.6, we have*

$$\tau \rtimes \pi^{\Sigma_0} \cong \tau^\vee \rtimes \pi^{\Sigma_0}.$$

Proof. Taking conjugation by elements in G^{Σ_0} , one can transform the inducing representation $\tau \otimes \pi^{\Sigma_0}$ to $\tau^\vee \otimes \pi^{\Sigma_0}$. Then the corollary follows from the irreducibility of both induced representations. \square

4. BASIC CASE AND GENERALIZATION

We describe the **basic case** as follows. Let us fix a self-dual unitary irreducible supercuspidal representation ρ of $GL(d_\rho)$. There exists

$$\{(\rho, A_2, B_2, \zeta_2), (\rho, A_1, B_1, \zeta_1)\} \subseteq Jord(\psi)$$

such that $A_2 \geq A_1, B_2 \geq B_1$, and $\zeta_1 = \zeta_2 = \zeta$. These two Jordan blocks are “separated” from the other blocks in $Jord_\rho(\psi)$. Moreover, let

$$Jord(\psi_-) = Jord(\psi) \setminus \{(\rho, A_2, B_2, \zeta_2), (\rho, A_1, B_1, \zeta_1)\},$$

we require ψ_- has discrete diagonal restriction. We can extend the natural order on $Jord(\psi_-)$ to $Jord(\psi)$ as follows

$$(\rho, A, B, \zeta) >_\psi (\rho, A', B', \zeta') \text{ if and only if } A \geq A'.$$

In particular,

$$(\rho, A_2, B_2, \zeta_2) >_\psi (\rho, A_1, B_1, \zeta_1).$$

For functions $\underline{l}(\rho, A, B, \zeta) \in [0, [(A - B + 1)/2]]$ and $\underline{\eta}(\rho, A, B, \zeta) \in \mathbb{Z}_2$ on $Jord(\psi)$, we denote

$$l_1 = \underline{l}(\rho, A_1, B_1, \zeta_1), \quad l_2 = \underline{l}(\rho, A_2, B_2, \zeta_2),$$

and

$$\eta_1 = \underline{\eta}(\rho, A_1, B_1, \zeta_1), \quad \eta_2 = \underline{\eta}(\rho, A_2, B_2, \zeta_2).$$

Lemma 4.1 (Mœglin). *In the basic case, suppose*

$$[A_2, B_2] = [A_1, B_1],$$

then $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if

$$\begin{cases} \eta_2 = (-1)^{A_1 - B_1} \eta_1 & \Rightarrow l_1 = l_2, \\ \eta_2 \neq (-1)^{A_1 - B_1} \eta_1 & \Rightarrow l_1 = l_2 = (A_1 - B_1 + 1)/2. \end{cases}$$

This lemma is proved in ([Mœg06a], Lemma 3.4). Since this result is fundamental for all the results that we are going to derive in this paper, we would like to give its proof in Appendix A. This lemma can also be easily generalized as follows.

Proposition 4.2. *In the basic case, if $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, then*

$$(4.1) \quad \begin{cases} \eta_2 = (-1)^{A_1 - B_1} \eta_1 & \Rightarrow A_2 - l_2 \geq A_1 - l_1, \quad B_2 + l_2 \geq B_1 + l_1, \\ \eta_2 \neq (-1)^{A_1 - B_1} \eta_1 & \Rightarrow B_2 + l_2 > A_1 - l_1. \end{cases}$$

Conversely, if (4.1) is satisfied, then $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, moreover

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta B_2 & \cdots & -\zeta A_2 \\ \vdots & & \vdots \\ \zeta(B_2 + l_2 - 1) & \cdots & -\zeta(A_2 - l_2 + 1) \end{pmatrix} \times \begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1 + l_1 - 1) & \cdots & -\zeta(A_1 - l_1 + 1) \end{pmatrix} \\ &\times \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - l_2, B_2 + l_2, 0, \eta_2, \zeta), (\rho, A_1 - l_1, B_1 + l_1, 0, \eta_1, \zeta)). \end{aligned}$$

Proof. We first show the necessity of the condition (4.1). So let us suppose $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, and we take the following two reduction steps.

- **First reduction:** we assume $A_2 > A_1$ and $l_2 \neq 0$.

Let us define ψ_{\gg} by shifting $(\rho, A_2, B_2, \zeta_2)$ to $(\rho, A_2 + T, B_2 + T, \zeta_2)$, such that ψ_{\gg} has discrete diagonal restriction and the natural order is the same as $>_{\psi}$. Then

$$\pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}, \underline{L}, \underline{\eta}) \hookrightarrow \langle \zeta(B_2 + T), \dots, -\zeta(A_2 + T) \rangle \rtimes \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{L}_{-}, \underline{\eta}_{-};$$

$$(\rho, A_2 + T - 1, B_2 + T + 1, l_2 - 1, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)).$$

Note

$$\text{Jac}_{(\rho, A_2+T, B_2+T, \zeta) \mapsto (\rho, A_2, B_2, \zeta)} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}, \underline{L}, \underline{\eta}) = \pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \underline{L}, \underline{\eta}) \neq 0.$$

So after applying $\text{Jac}_{(\rho, A_2+T, B_2+T, \zeta) \mapsto (\rho, A_2, B_2, \zeta)}$ to the full induced representation above, we have

$$\langle \zeta B_2, \dots, -\zeta A_2 \rangle \rtimes \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{L}_{-}, \underline{\eta}_{-}; (\rho, A_2 - 1, B_2 + 1, l_2 - 1, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)),$$

which is again nonzero. In particular,

$$\pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{L}_{-}, \underline{\eta}_{-}; (\rho, A_2 - 1, B_2 + 1, l_2 - 1, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \neq 0$$

By our assumption, $A_2 - 1 \geq A_1$, $B_2 + 1 > B_1$ and $l_2 - 1 \geq 0$, so we get

$$\begin{cases} \eta_2 = (-1)^{A_1 - B_1} \eta_1 & \Rightarrow (A_2 - 1) - (l_2 - 1) \geq A_1 - l_1, (B_2 + 1) + (l_2 - 1) \geq B_1 + l_1, \\ \eta_2 \neq (-1)^{A_1 - B_1} \eta_1 & \Rightarrow (B_2 + 1) + (l_2 - 1) > A_1 - l_1. \end{cases}$$

This gives the condition (4.1).

- **Second reduction:** we assume $B_2 > B_1$ and $l_1 \neq 0$.

We choose ψ_{\gg} as in the previous step. Then

$$\pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}, \underline{L}, \underline{\eta}) \hookrightarrow \langle \zeta B_1, \dots, -\zeta A_1 \rangle \rtimes \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{L}_{-}, \underline{\eta}_{-};$$

$$(\rho, A_2 + T, B_2 + T, l_2, \eta_2, \zeta), (\rho, A_1 - 1, B_1 + 1, l_1 - 1, \eta_1, \zeta)).$$

Note

$$\text{Jac}_{(\rho, A_2+T, B_2+T, \zeta) \mapsto (\rho, A_2, B_2, \zeta)} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}, \underline{L}, \underline{\eta}) = \pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \underline{L}, \underline{\eta}) \neq 0.$$

So after applying $\text{Jac}_{(\rho, A_2+T, B_2+T, \zeta) \mapsto (\rho, A_2, B_2, \zeta)}$ to the full induced representation above, we have

$$\langle \zeta B_1, \dots, -\zeta A_1 \rangle \rtimes \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{L}_{-}, \underline{\eta}_{-}; (\rho, A_2, B_2, l_2, \eta_2, \zeta), (\rho, A_1 - 1, B_1 + 1, l_1 - 1, \eta_1, \zeta)),$$

which is again nonzero. In particular,

$$\pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{L}_{-}, \underline{\eta}_{-}; (\rho, A_2, B_2, l_2, \eta_2, \zeta), (\rho, A_1 - 1, B_1 + 1, l_1 - 1, \eta_1, \zeta)) \neq 0$$

By our assumption, $A_2 > A_1 - 1$, $B_2 \geq B_1 + 1$ and $l_1 - 1 \geq 0$, so we get

$$\begin{cases} \eta_2 = (-1)^{A_1 - B_1} \eta_1 & \Rightarrow A_2 - l_2 \geq (A_1 - 1) - (l_1 - 1), B_2 + l_2 \geq (B_1 + 1) + (l_1 - 1), \\ \eta_2 \neq (-1)^{A_1 - B_1} \eta_1 & \Rightarrow B_2 + l_2 > (A_1 - 1) - (l_1 - 1). \end{cases}$$

This again gives the condition (4.1).

After these two steps, we are reduced to the following cases:

- **Case 1:** $A_2 = A_1$ and $B_2 = B_1$.

This case is treated in Lemma 4.1.

- **Case 2:** $A_2 = A_1$, $B_2 > B_1$ and $l_1 = 0$.

In this case, the condition (4.1) becomes

$$\eta_2 = (-1)^{A_1 - B_1} \eta_1 \text{ and } l_2 = 0.$$

Note

$$\pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \underline{L}, \underline{\eta}) = \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{L}_{-}, \underline{\eta}_{-}; (\rho, A_2, B_2, l_2, \eta_2, \zeta),$$

$$(\rho, A_1, B_2, 0, (-1)^{B_2-B_1}\eta_1, \zeta), (\rho, B_2 - 1, B_1, 0, \eta_1, \zeta)\Big).$$

Applying Lemma 4.1 to $(\rho, A_2, B_2, l_2, \eta_2, \zeta)$ and $(\rho, A_1, B_2, 0, (-1)^{B_2-B_1}\eta_1, \zeta)$, we get

$$\eta_2 = (-1)^{A_1-B_2} \cdot (-1)^{B_2-B_1}\eta_1 = (-1)^{A_1-B_1}\eta_1,$$

and $l_2 = 0$. This is exactly what we want.

- **Case 3:** $A_2 > A_1, B_2 = B_1$ and $l_2 = 0$.

In this case, the condition (4.1) becomes

$$\eta_2 = (-1)^{A_1-B_1}\eta_1 \text{ and } l_1 = 0.$$

Note

$$\begin{aligned} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &= \pi_{M, > \psi}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; (\rho, A_2 + T, A_1 + T + 1, 0, (-1)^{A_1-B_2+1}\eta_2, \zeta), \\ &\quad (\rho, A_1 + T, B_2 + T, 0, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)). \end{aligned}$$

Since

$$\text{Jac}_{(\rho, A_2+T, B_2+T, \zeta) \mapsto (\rho, A_2, B_2, \zeta)} = \text{Jac}_{(\rho, A_2+T, A_1+T+1, \zeta) \mapsto (\rho, A_2, A_1+1, \zeta)} \circ \text{Jac}_{(\rho, A_1+T, B_2+T, \zeta) \mapsto (\rho, A_1, B_2, \zeta)},$$

then

$$\begin{aligned} &\text{Jac}_{(\rho, A_1+T, B_2+T, \zeta) \mapsto (\rho, A_1, B_2, \zeta)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) = \\ &\pi_{M, > \psi}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; (\rho, A_2 + T, A_1 + T + 1, 0, (-1)^{A_1-B_2+1}\eta_2, \zeta), \\ &\quad (\rho, A_1, B_2, 0, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \neq 0. \end{aligned}$$

Applying Lemma 4.1 to $(\rho, A_1, B_2, 0, \eta_2, \zeta)$ and $(\rho, A_1, B_1, l_1, \eta_1, \zeta)$, we get exactly

$$\eta_2 = (-1)^{A_1-B_1}\eta_1 \text{ and } l_1 = 0.$$

- **Case 4:** $A_2 > A_1, B_2 > B_1$ and $l_2 = l_1 = 0$.

If $\eta_2 = (-1)^{A_1-B_1}\eta_1$, the condition is automatically satisfied.

If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$. We can suppose $B_2 \leq A_1$, and let $T = A_1 - B_2 + 1$. One observes

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) = \pi_{M, > \psi}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; (\rho, A_2 + T, B_1, 0, \eta_1, \zeta)).$$

So $\text{Jac}_{\zeta(A_1+1)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) = 0$. Therefore,

$$\text{Jac}_{(\rho, A_2+T, B_2+T, \zeta) \mapsto (\rho, A_2, B_2, \zeta)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) = 0.$$

Next we would like to show the sufficiency of condition (4.1) by computing $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ directly. We will take ψ_{\gg} to be defined as before.

- Suppose $l_1 = l_2 = 0$. If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, then $B_2 > A_1$ and there is nothing to prove. So let us also assume $\eta_2 = (-1)^{A_1-B_1}\eta_1$.

(1) $A_2 - B_2 \leq A_1 - B_1$.

$$\begin{aligned} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta(B_2 + T) & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(A_2 + T) & \cdots & -\zeta(A_1 - A_2 + B_2) \end{pmatrix} \\ &\rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; (\rho, A_1 - A_2 + B_2 - 1, B_1, 0, \eta_1, \zeta)). \end{aligned}$$

It is clear that $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$.

(2) $A_2 - B_2 > A_1 - B_1$.

$$\begin{aligned} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta(B_2 + T) & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_2 + A_1 - B_1 + T) & \cdots & -\zeta B_1 \end{pmatrix} \\ &\times \begin{pmatrix} \zeta(B_2 + A_1 - B_1 + T + 1) & \cdots & \zeta(B_2 + A_1 - B_1 + 2) \\ \vdots & & \vdots \\ \zeta(A_2 + T) & \cdots & \zeta(A_2 + 1) \end{pmatrix} \\ &\rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2, B_2 + A_1 - B_1 + 1, 0, -\eta_1, \zeta)). \end{aligned}$$

It is again clear that $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$.

- Suppose $l_1 \neq 0$ or $l_2 \neq 0$.

$$\begin{aligned} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta(B_2 + T) & \cdots & -\zeta(A_2 + T) \\ \vdots & & \vdots \\ \zeta(B_2 + l_2 - 1 + T) & \cdots & -\zeta(A_2 - l_2 + 1 + T) \end{pmatrix} \\ &\times \begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1 + l_1 - 1) & \cdots & -\zeta(A_1 - l_1 + 1) \end{pmatrix} \\ &\rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - l_2 + T, B_2 + l_2 + T, 0, \eta_2, \zeta), (\rho, A_1 - l_1, B_1 + l_1, 0, \eta_1, \zeta)). \end{aligned}$$

From our previous discussion, we know

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - l_2, B_2 + l_2, 0, \eta_2, \zeta), (\rho, A_1 - l_1, B_1 + l_1, 0, \eta_1, \zeta)) \neq 0,$$

so

$$\begin{aligned} \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - l_2 + T, B_2 + l_2 + T, 0, \eta_2, \zeta), (\rho, A_1 - l_1, B_1 + l_1, 0, \eta_1, \zeta)) \\ \hookrightarrow \begin{pmatrix} \zeta(B_2 + l_2 + T) & \cdots & \zeta(B_2 + l_2 + 1) \\ \vdots & & \vdots \\ \zeta(A_2 - l_2 + T) & \cdots & \zeta(A_2 - l_2 + 1) \end{pmatrix} \\ \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - l_2, B_2 + l_2, 0, \eta_2, \zeta), (\rho, A_1 - l_1, B_1 + l_1, 0, \eta_1, \zeta)). \end{aligned}$$

Therefore,

$$\begin{aligned} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta(B_2 + T) & \cdots & \zeta(B_2 + 1) \\ \vdots & & \vdots \\ \zeta(B_2 + l_2 - 1 + T) & \cdots & \zeta(B_2 + l_2) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta B_2 & \cdots & -\zeta(A_2 + T) \\ \vdots & & \vdots \\ \zeta(B_2 + l_2 - 1) & \cdots & -\zeta(A_2 - l_2 + 1 + T) \end{pmatrix}}_I \\ &\times \underbrace{\begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1 + l_1 - 1) & \cdots & -\zeta(A_1 - l_1 + 1) \end{pmatrix}}_{II} \times \underbrace{\begin{pmatrix} \zeta(B_2 + l_2 + T) & \cdots & \zeta(B_2 + l_2 + 1) \\ \vdots & & \vdots \\ \zeta(A_2 - l_2 + T) & \cdots & \zeta(A_2 - l_2 + 1) \end{pmatrix}}_{III} \\ &\rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - l_2, B_2 + l_2, 0, \eta_2, \zeta), (\rho, A_1 - l_1, B_1 + l_1, 0, \eta_1, \zeta)). \end{aligned}$$

Since $[\zeta B_2, -\zeta(A_2 + T)] \supseteq [\zeta B_1, -\zeta A_1]$, (I) and (II) are interchangeable. Also note $B_2 + l_2 + 1 > B_1 + l_1$, so we can interchange (II) and (III). It is clear that (I) and (III) are interchangeable

too. As a result,

$$\begin{aligned}
\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta(B_2 + T) & \cdots & \zeta(B_2 + 1) \\ \vdots & & \vdots \\ \zeta(B_2 + l_2 - 1 + T) & \cdots & \zeta(B_2 + l_2) \end{pmatrix} \\
&\times \underbrace{\begin{pmatrix} \zeta(B_2 + l_2 + T) & \cdots & \zeta(B_2 + l_2 + 1) \\ \vdots & & \vdots \\ \zeta(A_2 - l_2 + T) & \cdots & \zeta(A_2 - l_2 + 1) \end{pmatrix}}_{III} \times \underbrace{\begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1 + l_1 - 1) & \cdots & -\zeta(A_1 - l_1 + 1) \end{pmatrix}}_{II} \\
&\times \underbrace{\begin{pmatrix} \zeta B_2 & \cdots & -\zeta(A_2 + 1) & \cdots & -\zeta(A_2 + T) \\ \vdots & & \vdots & & \vdots \\ \zeta(B_2 + l_2 - 1) & \cdots & -\zeta(A_2 - l_2 + 2) & \cdots & -\zeta(A_2 - l_2 + 1 + T) \end{pmatrix}}_I \\
&\times \pi_{M, > \psi}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; (\rho, A_2 - l_2, B_2 + l_2, 0, \eta_2, \zeta), (\rho, A_1 - l_1, B_1 + l_1, 0, \eta_1, \zeta)).
\end{aligned}$$

By Proposition 3.6,

$$\begin{aligned}
&\underbrace{\begin{pmatrix} -\zeta(A_2 + 1) & \cdots & -\zeta(A_2 + T) \\ \vdots & & \vdots \\ -\zeta(A_2 - l_2 + 2) & \cdots & -\zeta(A_2 - l_2 + 1 + T) \end{pmatrix}}_{IV} \\
&\times \pi_{M, > \psi}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; (\rho, A_2 - l_2, B_2 + l_2, 0, \eta_2, \zeta), (\rho, A_1 - l_1, B_1 + l_1, 0, \eta_1, \zeta)).
\end{aligned}$$

is irreducible. So we can take the dual of (IV) (see Corollary 3.7). Moreover, (IV)[∨] is interchangeable with

$$\underbrace{\begin{pmatrix} \zeta B_2 & \cdots & -\zeta A_2 \\ \vdots & & \vdots \\ \zeta(B_2 + l_2 - 1) & \cdots & -\zeta(A_2 - l_2 + 1) \end{pmatrix}}_{I_{-}}$$

and (II). Then

$$\begin{aligned}
\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta(B_2 + T) & \cdots & \zeta(B_2 + 1) \\ \vdots & & \vdots \\ \zeta(B_2 + l_2 - 1 + T) & \cdots & \zeta(B_2 + l_2) \end{pmatrix} \\
&\times \underbrace{\begin{pmatrix} \zeta(B_2 + l_2 + T) & \cdots & \zeta(B_2 + l_2 + 1) \\ \vdots & & \vdots \\ \zeta(A_2 - l_2 + T) & \cdots & \zeta(A_2 - l_2 + 1) \end{pmatrix}}_{III} \\
&\times \underbrace{\begin{pmatrix} \zeta(A_2 - l_2 + 1 + T) & \cdots & \zeta(A_2 - l_2 + 2) \\ \vdots & & \vdots \\ \zeta(A_2 + T) & \cdots & \zeta(A_2 + 1) \end{pmatrix}}_{(IV)^{\vee}}
\end{aligned}$$

$$\begin{aligned}
& \times \underbrace{\begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1 + l_1 - 1) & \cdots & -\zeta(A_1 - l_1 + 1) \end{pmatrix}}_{II} \times \underbrace{\begin{pmatrix} \zeta B_2 & \cdots & -\zeta A_2 \\ \vdots & & \vdots \\ \zeta(B_2 + l_2 - 1) & \cdots & -\zeta(A_2 - l_2 + 1) \end{pmatrix}}_{I_-} \\
& \times \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - l_2, B_2 + l_2, 0, \eta_2, \zeta), (\rho, A_1 - l_1, B_1 + l_1, 0, \eta_1, \zeta) \right).
\end{aligned}$$

It follows $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, and

$$\begin{aligned}
\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) & \hookrightarrow \underbrace{\begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1 + l_1 - 1) & \cdots & -\zeta(A_1 - l_1 + 1) \end{pmatrix}}_{II} \times \underbrace{\begin{pmatrix} \zeta B_2 & \cdots & -\zeta A_2 \\ \vdots & & \vdots \\ \zeta(B_2 + l_2 - 1) & \cdots & -\zeta(A_2 - l_2 + 1) \end{pmatrix}}_{I_-} \\
& \times \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - l_2, B_2 + l_2, 0, \eta_2, \zeta), (\rho, A_1 - l_1, B_1 + l_1, 0, \eta_1, \zeta) \right).
\end{aligned}$$

Finally, one just needs to observe (II) and (I_-) are interchangeable. \square

Next we would like to generalize the basic case to the following situation. Suppose we can index $Jord_\rho(\psi)$ for each ρ such that $A_i \geq A_{i-1}$ and $B_i \geq B_{i-1}$. Moreover we can divide $Jord_\rho(\psi)$ into chunks of

$$(4.2) \quad \{(\rho, A_i, B_i, \zeta_i), (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1})\} \text{ with } \zeta_i = \zeta_{i-1}, \text{ or } \{(\rho, A_j, B_j, \zeta_j)\},$$

such that each of them is “separated” from the others in $Jord_\rho(\psi)$. We call this the **generalized basic case**. There is a natural order $>_\psi$ on $Jord_\rho(\psi)$, i.e.,

$$(\rho, A_i, B_i, \zeta_i) >_\psi (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

Proposition 4.3. *In the generalized basic case, $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if the condition (4.1) is satisfied for each chunk of pair $\{(\rho, A_i, B_i, \zeta_i), (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1})\}$ in (4.2) for all ρ .*

Proof. We will first prove the sufficiency of the nonvanishing condition by induction on the number of intersected pairs in $Jord(\psi)$. Let ρ be fixed. For $Jord_\rho(\psi)$, suppose n is the biggest integer such that $[A_n, B_n]$ and $[A_{n-1}, B_{n-1}]$ intersects. Let

$$Jord(\psi_-) = Jord(\psi) \setminus \{(\rho, A_n, B_n, \zeta_n)\}$$

By induction we can assume

$$\pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n, B_n + T_n, l_n, \eta_n, \zeta_n) \right) \neq 0$$

for the smallest T_n such that $[A_n + T_n, B_n + T_n]$ does not intersect with $[A_{n-1}, B_{n-1}]$. For those intersected pairs $\{(\rho, A_i, B_i, \zeta_i), (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1})\}$, we can put them apart by shifting $(\rho, A_i, B_i, \zeta_i)$ to $(\rho, A_i + T_i, B_i + T_i, \zeta_i)$ again for the smallest T_i . Let us write $T_j = 0$ for those $(\rho, A_j, B_j, \zeta_j)$ remained in $Jord_\rho(\psi)$. As a result we can get a parameter ψ_{\gg} dominating ψ with discrete diagonal restriction such that

$$(\rho, A_{\gg, i}, B_{\gg, i}, \zeta_i) = (\rho, A_i + T_i, B_i + T_i, \zeta_i).$$

Then

$$\begin{aligned}
\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) & \hookrightarrow \times_{i \neq n} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix} \times \mathcal{C}_{X^c} \\
& \times \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n, B_n + T_n, l_n, \eta_n, \zeta_n) \right),
\end{aligned}$$

where i is increasing. We would like to show

$$\text{Jac}_{(\rho, A_n+T_n, B_n+T_n, \zeta_n) \mapsto (\rho, A_n, B_n, \zeta_n)} \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n, B_n + T_n, l_n, \eta_n, \zeta_n) \right) \neq 0.$$

Note by our assumption,

$$\text{Jac}_{(\rho, A_n+T_n, B_n+T_n, \zeta_n) \mapsto (\rho, A_n, B_n, \zeta_n)} \pi_{M, > \psi}^{\Sigma_0} (\psi_{\gg}, \underline{l}, \underline{\eta}) \neq 0.$$

So after we apply the same Jacquet functor to the full induced representation above, we should get something nonzero. To compute this Jacquet module, one notes $B_n + 1 > A_i + T_i$ for $T_i \neq 0$, so it can only be

$$\begin{aligned} & \times_{i \neq n} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix} \times \mathcal{C}_{X^c} \rtimes \text{Jac}_{(\rho, A_n+T_n, B_n+T_n, \zeta_n) \mapsto (\rho, A_n, B_n, \zeta_n)} \\ & \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n, B_n + T_n, l_n, \eta_n, \zeta_n) \right) \neq 0. \end{aligned}$$

This gives what we want.

Next for the necessity of the nonvanishing condition, we can assume $\pi_{M, > \psi}^{\Sigma_0} (\psi, \underline{l}, \underline{\eta}) \neq 0$. We still fix ρ and choose a dominating parameter ψ_{\gg} with discrete diagonal restriction in the way as above. Then by definition

$$\pi_{M, > \psi}^{\Sigma_0} (\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \times_i \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix} \times \mathcal{C}_{X^c} \rtimes \pi_{M, > \psi}^{\Sigma_0} (\psi, \underline{l}, \underline{\eta}).$$

It is easy to see that those generalized segments in the induced representations are not linked. So we can change their orders in the induction. In particular, we can take any generalized segment to the front. As a result

$$\text{Jac}_{(\rho, A_i+T_i, B_i+T_i, \zeta_i) \mapsto (\rho, A_i, B_i, \zeta_i)} \pi_{M, > \psi}^{\Sigma_0} (\psi_{\gg}, \underline{l}, \underline{\eta}) \neq 0.$$

for any i . This gives the condition that we want with respect to ρ . By varying ρ , we prove the necessity of the condition. \square

Remark 4.4. Suppose a subset of Jordan blocks of $Jord_{\rho}(\psi)$ satisfies the condition in the generalized basic case, then we say the Jordan blocks in this set have “good shape”.

4.1. Some necessary conditions on nonvanishing. In this section, we want to use Proposition 4.2 to give some necessary conditions on the nonvanishing of $\pi_{M, > \psi}^{\Sigma_0} (\psi, \underline{l}, \underline{\eta})$ in general. Let us fix ρ and index the Jordan blocks in $Jord_{\rho}(\psi)$ such that

$$(\rho, A_i, B_i, \zeta_i) >_{\psi} (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

Let $(\rho, A_k, B_k, \zeta_k) >_{\psi} (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})$ be two adjacent blocks under the order $>_{\psi}$ and $\zeta_k = \zeta_{k-1}$.

Lemma 4.5. *Suppose $A_k \geq A_{k-1}$ and $B_k \geq B_{k-1}$. If $\pi_{M, > \psi}^{\Sigma_0} (\psi, \underline{l}, \underline{\eta}) \neq 0$, then $l_k, \eta_k, l_{k-1}, \eta_{k-1}$ satisfy the condition (4.1).*

Proof. Let ψ_{\gg} be a dominating parameter with discrete diagonal restriction. We also define $\psi^{(k)}$ from ψ_{\gg} by shifting $(\rho, A_i + T_i, B_i + T_i, \zeta_i)$ back to $(\rho, A_i, B_i, \zeta_i)$ for $i \leq k$. Then

$$\pi_{M, > \psi}^{\Sigma_0} (\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i < k-1} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix}}_I$$

$$\begin{aligned}
& \times \underbrace{\begin{pmatrix} \zeta_k(B_k + T_k) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k) & \cdots & \zeta_k(A_k + 1) \end{pmatrix}}_{II} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi^{(k)}, \underline{L}, \underline{\eta}) \\
& \hookrightarrow \times_{i < k-1} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix}}_I \\
& \times \underbrace{\begin{pmatrix} \zeta_k(B_k + T_k) & \cdots & \zeta_k(B_k + T_{k-1} + 1) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots & & \vdots \\ \zeta_k(A_{k-1} + T_k) & \cdots & \zeta_k(A_{k-1} + T_{k-1} + 1) & \cdots & \zeta_k(A_{k-1} + 1) \end{pmatrix}}_{II_1} \\
& \times \underbrace{\begin{pmatrix} \zeta_k(A_{k-1} + T_k + 1) & \cdots & \zeta_k(A_{k-1} + T_{k-1} + 2) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k) & \cdots & \zeta_k(A_k + T_{k-1} + 1) \end{pmatrix}}_{II_2} \\
& \times \underbrace{\begin{pmatrix} \zeta_k(A_{k-1} + T_{k-1} + 1) & \cdots & \zeta_k(A_{k-1} + 2) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_{k-1}) & \cdots & \zeta_k(A_k + 1) \end{pmatrix}}_{II_3} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi^{(k)}, \underline{L}, \underline{\eta}),
\end{aligned}$$

where i increases. Note (I) is interchangeable with (II_1) and (II_2) , and $B_k + T_{k-1} + 1 > A_i + T_i + 1$ for $i < k - 1$. As a result,

$$\text{Jac}_{(\rho, A_k + T_k, B_k + T_k, \zeta_k) \mapsto (\rho, A_k + T_{k-1}, B_k + T_{k-1}, \zeta_k)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{L}, \underline{\eta}) \neq 0.$$

Then by Proposition 4.2, $l_k, \eta_k, l_{k-1}, \eta_{k-1}$ satisfy the condition (4.1). \square

Lemma 4.6. Suppose $[A_k, B_k] \supseteq [A_{k-1}, B_{k-1}]$. If $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{L}, \underline{\eta}) \neq 0$, then $l_k, \eta_k, l_{k-1}, \eta_{k-1}$ satisfy the following condition:

$$(4.3) \quad \begin{cases} \eta_k = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} & \Rightarrow 0 \leq l_k - l_{k-1} \leq (A_k - B_k) - (A_{k-1} - B_{k-1}), \\ \eta_k \neq (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} & \Rightarrow l_k + l_{k-1} > A_{k-1} - B_{k-1}. \end{cases}$$

Proof. Let ψ_{\gg} be a dominating parameter with discrete diagonal restriction. We also define $\psi^{(k)}$ from ψ_{\gg} by shifting $(\rho, A_i + T_i, B_i + T_i, \zeta_i)$ back to $(\rho, A_i, B_i, \zeta_i)$ for $i \leq k$. Then

$$\begin{aligned}
& \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{L}, \underline{\eta}) \hookrightarrow \times_{i < k-1} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix}}_I \\
& \times \underbrace{\begin{pmatrix} \zeta_k(B_k + T_k) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k) & \cdots & \zeta_k(A_k + 1) \end{pmatrix}}_{II} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi^{(k)}, \underline{L}, \underline{\eta}),
\end{aligned}$$

where i increases. Note (I) and (II) are interchangeable due to $[A_k + 1, B_k + 1] \supseteq [A_{k-1} + 1, B_{k-1} + 1]$. Since $B_k + T_{k-1} + 1 > A_i + T_i + 1$ for $i < k - 1$, we have

$$\text{Jac}_{(\rho, A_k + T_k, B_k + T_k, \zeta_k) \mapsto (\rho, A_k + T_{k-1}, B_k + T_{k-1}, \zeta_k)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \neq 0.$$

In particular,

$$\text{Jac}_{(\rho, A_k + T_k, B_k + T_k, \zeta_k) \mapsto (\rho, A_k + T_{k-1} + B_{k-1} - B_k, B_{k-1} + T_{k-1}, \zeta_k)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \neq 0.$$

By Proposition 4.2, the condition (4.1) is satisfied for $(\rho, A_k + T_{k-1} + B_{k-1} - B_k, B_{k-1} + T_{k-1}, l_k, \eta_k, \zeta_k)$ and $(\rho, A_{k-1} + T_{k-1}, B_{k-1} + T_{k-1}, l_{k-1}, \eta_{k-1}, \zeta_{k-1})$, i.e.,

- If $\eta_k = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1}$, then

$$\begin{cases} (A_k + T_{k-1} + B_{k-1} - B_k) - l_k \geq (A_{k-1} + T_{k-1}) - l_{k-1} \Rightarrow l_k - l_{k-1} \leq (A_k - B_k) - (A_{k-1} - B_{k-1}), \\ (B_{k-1} + T_{k-1}) + l_k \geq (B_{k-1} + T_{k-1}) + l_{k-1} \Rightarrow l_k - l_{k-1} \geq 0. \end{cases}$$

- If $\eta_k \neq (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1}$, then

$$(B_{k-1} + T_{k-1}) + l_k > (A_{k-1} + T_{k-1}) - l_{k-1} \Rightarrow l_k + l_{k-1} > A_{k-1} - B_{k-1}.$$

This finishes the proof. \square

Lemma 4.7. *Suppose $[A_k, B_k] \subseteq [A_{k-1}, B_{k-1}]$. If $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, then $l_k, l_{k-1}, \eta_k, \eta_{k-1}$ satisfy the following condition:*

$$(4.4) \quad \begin{cases} \eta_k = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} & \Rightarrow 0 \leq l_{k-1} - l_k \leq (A_{k-1} - B_{k-1}) - (A_k - B_k), \\ \eta_k \neq (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} & \Rightarrow l_k + l_{k-1} > A_k - B_k. \end{cases}$$

Proof. Let ψ_{\gg} be a dominating parameter with discrete diagonal restriction. We also define $\psi^{(k)}$ from ψ_{\gg} by shifting $(\rho, A_i + T_i, B_i + T_i, \zeta_i)$ back to $(\rho, A_i, B_i, \zeta_i)$ for $i \leq k$. Then

$$\begin{aligned} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \times_{i < k-1} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix}}_I \\ &\quad \times \underbrace{\begin{pmatrix} \zeta_k(B_k + T_k) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k) & \cdots & \zeta_k(A_k + 1) \end{pmatrix}}_{II} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi^{(k)}, \underline{l}, \underline{\eta}), \end{aligned}$$

where i increases. Note (I) and (II) are interchangeable due to $[A_k + 1, B_k + 1] \subseteq [A_{k-1} + 1, B_{k-1} + 1]$. Since $B_k + T_{k-1} + 1 > A_i + T_i + 1$ for $i < k - 1$, we have

$$\text{Jac}_{(\rho, A_k + T_k, B_k + T_k, \zeta_k) \mapsto (\rho, A_k + T_{k-1}, B_k + T_{k-1}, \zeta_k)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \neq 0.$$

In particular,

$$\text{Jac}_{(\rho, A_k + T_k, B_k + T_k, \zeta_k) \mapsto (\rho, A_{k-1} + T_{k-1}, B_k + T_{k-1} + A_{k-1} - A_k, \zeta_k)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \neq 0.$$

By Proposition 4.2, the condition (4.1) is satisfied for $(\rho, A_{k-1} + T_{k-1}, B_k + T_{k-1} + A_{k-1} - A_k, l_k, \eta_k, \zeta_k)$ and $(\rho, A_{k-1} + T_{k-1}, B_{k-1} + T_{k-1}, l_{k-1}, \eta_{k-1}, \zeta_{k-1})$, i.e.,

- If $\eta_k = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1}$, then

$$\begin{cases} (A_{k-1} + T_{k-1}) - l_k \geq (A_{k-1} + T_{k-1}) - l_{k-1} \Rightarrow l_{k-1} - l_k \geq 0, \\ (B_k + T_{k-1} + A_{k-1} - A_k) + l_k \geq (B_{k-1} + T_{k-1}) + l_{k-1} \Rightarrow l_{k-1} - l_k \leq (A_{k-1} - B_{k-1}) - (A_k - B_k). \end{cases}$$

- If $\eta_k \neq (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$, then

$$(B_k + T_{k-1} + A_{k-1} - A_k) + l_k > (A_{k-1} + T_{k-1}) - l_{k-1} \Rightarrow l_k + l_{k-1} > A_k - B_k.$$

This finishes the proof. □

5. CHANGE OF ORDER FORMULAS

For $\psi = \psi_p \in \bar{\Psi}(G)$, we want to show how Mœglin's parametrization of elements in $\Pi_{\psi}^{\Sigma_0}$ changes as we change the order $>_{\psi}$. So we will fix an admissible order $>_{\psi}$ and we also fix a self-dual unitary irreducible supercuspidal representation ρ of $GL(d_{\rho})$. We index the Jordan blocks in $Jord_{\rho}(\psi)$ such that

$$(\rho, A_i, B_i, \zeta_i) >_{\psi} (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

Let $(\rho, A_k, B_k, \zeta_k) >_{\psi} (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})$ be two adjacent blocks under the order $>_{\psi}$. We denote by $>_{\psi}'$ the order obtained from $>_{\psi}$ by switching $(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})$ and $(\rho, A_k, B_k, \zeta_k)$. And we assume $>_{\psi}'$ is also admissible. Then we are in the following two cases.

5.1. Case $\zeta_k = \zeta_{k-1}$. In this case, we can assume without loss of generality that $[A_k, B_k] \supseteq [A_{k-1}, B_{k-1}]$. For functions $\underline{l}(\rho, A, B, \zeta) \in [0, [(A - B + 1)/2]]$ and $\underline{\eta}(\rho, A, B, \zeta) \in \mathbb{Z}_2$ on $Jord(\psi)$, we denote

$$l_k = \underline{l}(\rho, A_k, B_k, \zeta_k), \quad l_{k-1} = \underline{l}(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}),$$

and

$$\eta_k = \underline{\eta}(\rho, A_k, B_k, \zeta_k), \quad \eta_{k-1} = \underline{\eta}(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}).$$

From $(\underline{l}, \underline{\eta})$ satisfying (4.3), we want to construct another pair $(\underline{l}', \underline{\eta}')$ such that

$$\underline{l}'(\cdot) = \underline{l}(\cdot) \text{ and } \underline{\eta}'(\cdot) = \underline{\eta}(\cdot)$$

over $Jord(\psi) \setminus \{(\rho, A_k, B_k, \zeta_k), (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})\}$. Let us denote

$$l'_k = \underline{l}'(\rho, A_k, B_k, \zeta_k), \quad l'_{k-1} = \underline{l}'(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}),$$

and

$$\eta'_k = \underline{\eta}'(\rho, A_k, B_k, \zeta_k), \quad \eta'_{k-1} = \underline{\eta}'(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}).$$

Then we define $l'_k, l'_{k-1}, \eta'_k, \eta'_{k-1}$ according to the following formulas.

- If $\eta_k \neq (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$, then $\eta'_{k-1} = (-1)^{A_k-B_k}\eta'_k$ and

$$\begin{cases} l_{k-1} = l'_{k-1} \\ l_k - l'_k = (A_{k-1} - B_{k-1} - 2l_{k-1}) + 1 \\ \eta'_{k-1} = (-1)^{A_k-B_k}\eta_{k-1} \end{cases}$$

- If $\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and

$$l_k - l_{k-1} < (A_k - B_k)/2 - (A_{k-1} - B_{k-1}) + l_{k-1},$$

then $\eta'_{k-1} \neq (-1)^{A_k-B_k}\eta'_k$ and

$$\begin{cases} l_{k-1} = l'_{k-1} \\ l'_k - l_k = (A_{k-1} - B_{k-1} - 2l_{k-1}) + 1 \\ \eta'_{k-1} = (-1)^{A_k-B_k}\eta_{k-1} \end{cases}$$

- If $\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and

$$l_k - l_{k-1} \geq (A_k - B_k)/2 - (A_{k-1} - B_{k-1}) + l_{k-1},$$

then $\eta'_{k-1} = (-1)^{A_k-B_k}\eta'_k$ and

$$\begin{cases} l_{k-1} = l'_{k-1} \\ (l'_k - l'_{k-1}) + (l_k - l_{k-1}) = (A_k - B_k) - (A_{k-1} - B_{k-1}) \\ \eta'_{k-1} = (-1)^{A_k-B_k}\eta_{k-1} \end{cases}$$

One can check $(\underline{l}', \underline{\eta}')$ satisfies (4.4). We denote this transformation by S^+ . We can also define its “inverse” S^- , namely we start with any $(\underline{l}', \underline{\eta}')$ satisfying (4.4), and we define $l_k, l_{k-1}, \eta_k, \eta_{k-1}$ according to the following formulas.

- If $\eta'_{k-1} \neq (-1)^{A_k-B_k} \eta'_k$, then $\eta_k = (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1}$ and

$$\begin{cases} l_{k-1} = l'_{k-1} \\ l'_k - l_k = (A_{k-1} - B_{k-1} - 2l_{k-1}) + 1 \\ \eta'_{k-1} = (-1)^{A_k-B_k} \eta_{k-1} \end{cases}$$

- If $\eta'_{k-1} = (-1)^{A_k-B_k} \eta'_k$ and

$$l'_k - l'_{k-1} < (A_k - B_k)/2 - (A_{k-1} - B_{k-1}) + l'_{k-1},$$

then $\eta_k \neq (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1}$ and

$$\begin{cases} l_{k-1} = l'_{k-1} \\ l_k - l'_k = (A_{k-1} - B_{k-1} - 2l_{k-1}) + 1 \\ \eta'_{k-1} = (-1)^{A_k-B_k} \eta_{k-1} \end{cases}$$

- If $\eta'_{k-1} = (-1)^{A_k-B_k} \eta'_k$ and

$$l'_k - l'_{k-1} \geq (A_k - B_k)/2 - (A_{k-1} - B_{k-1}) + l'_{k-1},$$

then $\eta_k = (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1}$ and

$$\begin{cases} l_{k-1} = l'_{k-1} \\ (l'_k - l'_{k-1}) + (l_k - l_{k-1}) = (A_k - B_k) - (A_{k-1} - B_{k-1}) \\ \eta'_{k-1} = (-1)^{A_k-B_k} \eta_{k-1} \end{cases}$$

One can also check $(\underline{l}, \underline{\eta})$ satisfies (4.3). Moreover, we have

$$S^- \circ S^+(\underline{l}, \underline{\eta}) \sim_{\Sigma_0} (\underline{l}, \underline{\eta}),$$

and

$$S^+ \circ S^-(\underline{l}', \underline{\eta}') \sim_{\Sigma_0} (\underline{l}', \underline{\eta}').$$

So S^+ (resp. S^-) induces a bijection between $(\underline{l}, \underline{\eta})$ satisfying (4.3) and $(\underline{l}', \underline{\eta}')$ satisfying (4.4) modulo the equivalence relation \sim_{Σ_0} on both sides.

Proposition 5.1. *Suppose $(\underline{l}', \underline{\eta}') = S^+(\underline{l}, \underline{\eta})$, then*

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, >_{\psi'}}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}').$$

Let ψ_{\gg} be a dominating parameter of ψ such that $Jord_\rho(\psi_{\gg}) = Jord_\rho(\psi)$, and the Jordan blocks in $Jord_{\rho'}(\psi_{\gg})$ are disjoint for $\rho' \neq \rho$. Then

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \text{Jac}_{X^c} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}),$$

and

$$\pi_{M, >_{\psi'}}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') = \text{Jac}_{X^c} \pi_{M, >_{\psi'}}^{\Sigma_0}(\psi_{\gg}, \underline{l}', \underline{\eta}').$$

So it suffices to prove the proposition for such ψ_{\gg} . Therefore, in the following discussions of the proof of this proposition, we will always assume the Jordan blocks in $Jord_{\rho'}(\psi)$ are disjoint for $\rho' \neq \rho$, and if we choose some dominating ψ_{\gg} of ψ , we will always assume $Jord_{\rho'}(\psi_{\gg}) = Jord_{\rho'}(\psi)$ for $\rho' \neq \rho$.

5.1.1. *Reduction.* Let $(\underline{l}', \underline{\eta}') = S^+(\underline{l}, \underline{\eta})$. We want to reduce the proposition to the following case:

$$(5.1) \quad (\rho, A_i, B_i, \zeta_i) \gg (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}) \text{ for } i \neq k, (\rho, A_k, B_k, \zeta_k) \gg (\rho, A_{k-2}, B_{k-2}, \zeta_{k-2}) \text{ and } 0.$$

We will do this in two steps. First we will reduce it to the case:

$$(5.2) \quad (\rho, A_i, B_i, \zeta_i) \gg (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}) \text{ for } i > k, (\rho, A_k, B_k, \zeta_k) \gg \cup_{j=1}^{k-2} \{(\rho, A_j, B_j, \zeta_j)\} \text{ and } 0.$$

Let us choose a dominating parameter ψ_{\gg} with respect to $>_{\psi}$ such that $T_i = 0$ for $i < k-1$,

$$(\rho, A_i + T_i, B_i + T_i, \zeta_i) \gg (\rho, A_{i-1} + T_{i-1}, B_{i-1} + T_{i-1}, \zeta_{i-1}) \text{ for } i \geq k$$

and

$$(\rho, A_k + T_{k-1}, B_k + T_{k-1}, \zeta_k) \gg \cup_{j=1}^{k-2} \{(\rho, A_j, B_j, \zeta_j)\} \text{ and } 0.$$

From ψ_{\gg} , we can obtain a dominating parameter ψ'_{\gg} with respect to $>_{\psi'}$ such that $T'_i = T_i$ for $i \neq k, k-1$, and $T'_k = T_{k-1}, T'_{k-1} = T_k$. Let us also denote T_{k-1} by T , and construct ψ^T_{\gg} from ψ_{\gg} by changing T_k to T . Let $\psi^{(k)}_{\gg}$ be obtained from ψ_{\gg} by changing T_k, T_{k-1} to zero.

Suppose $\pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, then

$$\begin{aligned} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix} \times \begin{pmatrix} \zeta_k(B_k + T_k) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k) & \cdots & \zeta_k(A_k + 1) \end{pmatrix} \\ &\rtimes \pi_{M, >_{\psi}}^{\Sigma_0}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}), \end{aligned}$$

where the two generalized segments are interchangeable. So $\pi_{M, >_{\psi}}^{\Sigma_0}(\psi^T_{\gg}, \underline{l}, \underline{\eta}) \neq 0$, and

$$\begin{aligned} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi^T_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta_k(B_k + T) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T) & \cdots & \zeta_k(A_k + 1) \end{pmatrix} \\ &\times \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix} \rtimes \pi_{M, >_{\psi}}^{\Sigma_0}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}). \end{aligned}$$

By (5.2),

$$\pi_{M, >_{\psi'}}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') = \pi_{M, >_{\psi}}^{\Sigma_0}(\psi^T_{\gg}, \underline{l}, \underline{\eta}) \neq 0.$$

Then

$$\begin{aligned} \pi_{M, >_{\psi'}}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') &\hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + T + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + T + 1) \end{pmatrix} \rtimes \pi_{M, >_{\psi}}^{\Sigma_0}(\psi^T_{\gg}, \underline{l}, \underline{\eta}) \\ &\hookrightarrow \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + T + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + T + 1) \end{pmatrix}}_I \\ &\times \underbrace{\begin{pmatrix} \zeta_k(B_k + T) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T) & \cdots & \zeta_k(A_k + 1) \end{pmatrix}}_{II} \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + T) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix}}_{III} \\ &\rtimes \pi_{M, >_{\psi}}^{\Sigma_0}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}). \end{aligned}$$

We can interchange (II) and (III). If $B_{k-1} \neq B_k$, then $\text{Jac}_{\zeta_{k-1}(B_{k-1}+T)} \pi_{M, >_\psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') = 0$. So we can “combine” (I) and (III), i.e.,

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') &\hookrightarrow \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix}}_{I+III} \times \underbrace{\begin{pmatrix} \zeta_k(B_k + T) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T) & \cdots & \zeta_k(A_k + 1) \end{pmatrix}}_{II} \\ &\rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}). \end{aligned}$$

If $B_{k-1} = B_k$, let us write

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') &\hookrightarrow \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + T + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + T + 1) \end{pmatrix}}_I \\ &\times \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T) \\ \vdots \\ \zeta_{k-1}(A_{k-1} + T) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + T - 1) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T - 1) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix}}_{III_-} \\ &\times \underbrace{\begin{pmatrix} \zeta_k(B_k + T) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T) & \cdots & \zeta_k(A_k + 1) \end{pmatrix}}_{II} \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}). \end{aligned}$$

There exists an irreducible constituent σ of

$$\underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + T + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + T + 1) \end{pmatrix}}_I \times \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T) \\ \vdots \\ \zeta_{k-1}(A_{k-1} + T) \end{pmatrix}$$

such that

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') &\hookrightarrow \sigma \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + T - 1) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T - 1) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix}}_{III_-} \\ &\times \underbrace{\begin{pmatrix} \zeta_k(B_k + T) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T) & \cdots & \zeta_k(A_k + 1) \end{pmatrix}}_{II} \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}). \end{aligned}$$

Suppose $\text{Jac}_{\zeta_{k-1}(B_{k-1}+T)} \sigma \neq 0$, then $\text{Jac}_{\zeta_{k-1}(B_{k-1}+T)} \sigma$ is contained in

$$\underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + T + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + T + 1) \end{pmatrix}}_I \times \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T + 1) \\ \vdots \\ \zeta_{k-1}(A_{k-1} + T) \end{pmatrix}$$

which is irreducible. So

$$\sigma \hookrightarrow \rho|_{\zeta_{k-1}(B_{k-1}+T)} \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+T+1) \end{pmatrix}}_I \times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots \\ \zeta_{k-1}(A_{k-1}+T) \end{pmatrix}.$$

Hence

$$\begin{aligned} \pi_{M, >'_\psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') &\hookrightarrow \rho|_{\zeta_{k-1}(B_{k-1}+T)} \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+T+1) \end{pmatrix}}_I \\ &\times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots \\ \zeta_{k-1}(A_{k-1}+T) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta_k(B_k+T) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T) & \cdots & \zeta_k(A_k+1) \end{pmatrix}}_{II} \\ &\times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T-1) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T-1) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}}_{III_-} \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}, \underline{\eta}). \end{aligned}$$

If $A_k = A_{k-1}$, then $[A_k, B_k] = [A_{k-1}, B_{k-1}]$, and there is nothing to prove. So we can assume $A_k > A_{k-1}$. Then

$$\begin{aligned} \pi_{M, >'_\psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') &\hookrightarrow \rho|_{\zeta_{k-1}(B_{k-1}+T)} \times \begin{pmatrix} \zeta_k(B_k+T) \\ \vdots \\ \zeta_k(A_k+T) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+T+1) \end{pmatrix}}_I \\ &\times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots \\ \zeta_{k-1}(A_{k-1}+T) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta_k(B_k+T-1) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T-1) & \cdots & \zeta_k(A_k+1) \end{pmatrix}}_{II_-} \\ &\times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T-1) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T-1) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}}_{III_-} \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}). \end{aligned}$$

As a result, we have

$$\text{Jac}_{\zeta_{k-1}(B_{k-1}+T), \zeta_k(B_k+T)} \pi_{M, >'_\psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') \neq 0,$$

which is impossible. Therefore, we must have $\text{Jac}_{\zeta_{k-1}(B_{k-1}+T)} \sigma = 0$, and hence

$$\sigma = \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+T) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+T) \end{pmatrix}}_{I_+}$$

In this case,

$$\text{Jac}_{\zeta_{k-1}(B_{k-1}+T-1)} \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}') = 0.$$

So we again have

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') &\hookrightarrow \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix}}_{(I+III)} \times \underbrace{\begin{pmatrix} \zeta_k(B_k + T) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T) & \cdots & \zeta_k(A_k + 1) \end{pmatrix}}_{II} \\ &\rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}, \underline{\eta}). \end{aligned}$$

Since $[\zeta_{k-1}(A_{k-1} + T'_{k-1}), \zeta_{k-1}(A_{k-1} + 1)] \supseteq [\zeta_k(A_k + T), \zeta_k(A_k + 1)]$, we can interchange $(I + III)$ and (II) . Therefore,

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}', \underline{\eta}') = \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}, \underline{\eta}).$$

After applying $\circ_{i>k} \text{Jac}_{(\rho, A_i+T_i, B_i+T_i, \zeta_i) \mapsto (\rho, A_i, B_i, \zeta_i)}$ to both sides, we get

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}').$$

Secondly, we want to further reduce it to (5.1). So let us assume we are in case (5.2). We can choose a dominating parameter ψ_{\gg} with discrete diagonal restriction so that $T_i = 0$ for $i > k$ and $i = k - 1$. We also require

$$(\rho, A_i + T_i, B_i + T_i, \zeta_i) \gg (\rho, A_{i-1} + T_{i-1}, B_{i-1} + T_{i-1}, \zeta_{i-1}) \text{ for } i < k,$$

and

$$(\rho, A_k, B_k, \zeta_k) \gg (\rho, A_{k-2} + T_{k-2}, B_{k-2} + T_{k-2}, \zeta_{k-2}) \text{ and } 0.$$

Suppose $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, then

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \times_{i < k-1} \underbrace{\begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix}}_{I_i} \times \underbrace{\begin{pmatrix} \zeta_k(B_k + T_k) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k) & \cdots & \zeta_k(A_k + 1) \end{pmatrix}}_{II} \\ &\rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}), \end{aligned}$$

where i increases. Since $B_k + 1 > A_i + T_i + 1$ for $i < k - 1$, we can interchange (II) with (I_i) . Let $\psi_{\gg}^{(k)}$ be obtained from ψ_{\gg} by changing T_k to zero. Then

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i < k-1} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix} \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}),$$

By (5.1),

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}, \underline{\eta}) = \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}', \underline{\eta}') \neq 0.$$

Since

$$\text{Jac}_{(\rho, A_k+T_k, B_k+T_k, \zeta_k) \mapsto (\rho, A_k, B_k, \zeta_k)}$$

commutes with

$$\circ_{i < k-1} \text{Jac}_{(\rho, A_i+T_i, B_i+T_i, \zeta_k) \mapsto (\rho, A_i, B_i, \zeta_i)},$$

we have

$$\circ_{i < k-1} \text{Jac}_{(\rho, A_i+T_i, B_i+T_i, \zeta_k) \mapsto (\rho, A_i, B_i, \zeta_i)} \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}, \underline{\eta}) = \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$$

Similarly,

$$\circ_{i < k-1} \text{Jac}_{(\rho, A_i+T_i, B_i+T_i, \zeta_k) \mapsto (\rho, A_i, B_i, \zeta_i)} \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}', \underline{\eta}') = \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}')$$

So

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}').$$

This finishes our reduction step.

5.1.2. *Critical case.* From the previous reduction, we can now assume (5.1):

$$(\rho, A_i, B_i, \zeta_i) \gg (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}) \text{ for } i \neq k, (\rho, A_k, B_k, \zeta_k) \gg (\rho, A_{k-2}, B_{k-2}, \zeta_{k-2}) \text{ and } 0.$$

In this critical case, we can actually get the nonvanishing condition.

Lemma 5.2. *Suppose we are in case (5.1).*

(1) $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if

$$\begin{cases} \eta_k = (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1} & \Rightarrow 0 \leq l_k - l_{k-1} \leq (A_k - B_k) - (A_{k-1} - B_{k-1}), \\ \eta_k \neq (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1} & \Rightarrow l_k + l_{k-1} > A_{k-1} - B_{k-1}. \end{cases}$$

(2) $\pi_{M, >'_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') \neq 0$ if and only if

$$\begin{cases} \eta'_{k-1} = (-1)^{A_k-B_k} \eta'_k & \Rightarrow 0 \leq l'_k - l'_{k-1} \leq (A_k - B_k) - (A_{k-1} - B_{k-1}), \\ \eta'_{k-1} \neq (-1)^{A_k-B_k} \eta'_k & \Rightarrow l'_k + l'_{k-1} > A_{k-1} - B_{k-1}. \end{cases}$$

Proof. We will only show (1), and (2) is similar. One first notes the necessity of the nonvanishing condition has been shown in Lemma 4.6, so we get an upper bound for the size of the packet $|\Pi_\psi^{\Sigma_0}|$. In fact we can also get a lower bound for it. Let us define ψ^* by changing (ρ, A_k, B_k, ζ) to $(\rho, A_{k-1}, B_k - A_k + A_{k-1}, \zeta)$. Then the functor $\text{Jac}_{(\rho, A_k, B_k, \zeta) \mapsto (\rho, A_{k-1}, B_k - A_k + A_{k-1}, \zeta)}$ induces a surjection from $\Pi_\psi^{\Sigma_0}$ to $\Pi_{\psi^*}^{\Sigma_0}$:

$$\pi_{M, >_{\psi^*}}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \text{Jac}_{(\rho, A_k, B_k, \zeta) \mapsto (\rho, A_{k-1}, B_k - A_k + A_{k-1}, \zeta)} \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}).$$

So $|\Pi_{\psi^*}^{\Sigma_0}| < |\Pi_\psi^{\Sigma_0}|$. By Proposition 4.2, we have $\pi_{M, >'_\psi}^{\Sigma_0}(\psi^*, \underline{l}', \underline{\eta}') \neq 0$ if and only if

$$\begin{cases} \eta'_{k-1} = (-1)^{A_k-B_k} \eta'_k & \Rightarrow 0 \leq l'_k - l'_{k-1} \leq (A_{k-1} - (B_k - A_k + A_{k-1})) - (A_{k-1} - B_{k-1}), \\ \eta'_{k-1} \neq (-1)^{A_k-B_k} \eta'_k & \Rightarrow l'_k + l'_{k-1} > A_{k-1} - B_{k-1}. \end{cases}$$

Comparing this condition with the necessary condition for $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, one can easily see that $|\Pi_{\psi^*}^{\Sigma_0}|$ is equal to the upper bound for $|\Pi_\psi^{\Sigma_0}|$. Therefore, $|\Pi_\psi^{\Sigma_0}|$ must be equal to its upper bound, i.e., the necessary condition for $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$ is also sufficient. \square

Now we begin to prove the change of order formula in this case. Let us define ψ_- by

$$Jord(\psi_-) = Jord(\psi) \setminus \{(\rho, A_k, B_k, \zeta_k), (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})\},$$

then ψ_- has discrete diagonal restriction. Let $\zeta = \zeta_k = \zeta_{k-1}$. We are going to break the proof into four steps.

Step One: We want to show if $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, >'_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') \neq 0$, then

$$\underline{l}'(\cdot) = \underline{l}(\cdot) \text{ and } \underline{\eta}'(\cdot) = \underline{\eta}(\cdot)$$

over $Jord(\psi_-)$.

- Suppose $\underline{l}(\cdot) = 0$ over $Jord(\psi_-)$. We can define ψ_e by

$$Jord(\psi_e) := \cup_{i \neq k, k-1; C_i \in [A_i, B_i]} \{(\rho, C_i, C_i, \zeta_i)\} \cup \{(\rho, A_k, B_k, \zeta_k), (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})\}.$$

And we define ψ_{e-} by removing $(\rho, A_k, B_k, \zeta_k), (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})$. From $(\underline{l}, \underline{\eta})$, we obtain $(\underline{l}_e, \underline{\eta}_e)$ such that

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, >_\psi}^{\Sigma_0}(\psi_e, \underline{l}_e, \underline{\eta}_e).$$

Similarly, we can assume

$$\pi_{M, >'_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') = \pi_{M, >'_\psi}^{\Sigma_0}(\psi_e, \underline{l}'_e, \underline{\eta}'_e).$$

By computing $\varepsilon_{\psi_e}^{M/W}$ with respect to $>_\psi$ and $>'_\psi$, one finds if $\pi_{M, >_\psi}^{\Sigma_0}(\psi_e, \underline{l}_e, \underline{\eta}_e) = \pi_{M, >'_\psi}^{\Sigma_0}(\psi_e, \underline{l}'_e, \underline{\eta}'_e)$, then

$$\underline{l}_e(\cdot) = \underline{l}'_e(\cdot) = 0 \text{ and } \underline{\eta}'_e(\cdot) = \underline{\eta}_e(\cdot)$$

over $Jord(\psi_{e-})$. Therefore,

$$\underline{l}'(\cdot) = \underline{l}(\cdot) = 0 \text{ and } \underline{\eta}'(\cdot) = \underline{\eta}(\cdot)$$

over $Jord(\psi_-)$.

- Suppose $\underline{l}(\rho, A_i, B_i, \zeta_i) \neq 0$ for some $i \neq k, k-1$. Let $(\psi_0, \underline{l}_0)$ be obtained from (ψ, \underline{l}) by changing $(\rho, A_i, B_i, \zeta_i)$ to $(\rho, A_i - l_i, B_i + l_i, \zeta_i)$ for all $i \neq k, k-1$, and letting $\underline{l}_0(\rho, A_i - l_i, B_i + l_i, \zeta_i) = 0$. Then

$$\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow (\times_{i \neq k, k-1} \tau_i) \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_0, \underline{l}_0, \underline{\eta}),$$

as the unique irreducible subrepresentation, where

$$\tau_i = \begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i + l_i - 1) & \cdots & -\zeta_i(A_i - l_i + 1) \end{pmatrix}.$$

Suppose $\pi_{M, > \psi}^{\Sigma_0}(\psi_0, \underline{l}_0, \underline{\eta}) = \pi_{M, > \psi}^{\Sigma_0}(\psi_0, \underline{l}'_0, \underline{\eta}')$. We know from the previous discussion that

$$\underline{l}'_0(\cdot) = \underline{l}_0(\cdot) = 0 \text{ and } \underline{\eta}'(\cdot) = \underline{\eta}(\cdot)$$

over $Jord(\psi_0) \setminus \{(\rho, A_k, B_k, \zeta_k), (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})\}$. From \underline{l}'_0 we can obtain \underline{l}' such that $l'_i = l_i$ for $i \neq k, k-1$. Then

$$\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') \hookrightarrow (\times_{i \neq k, k-1} \tau_i) \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_0, \underline{l}'_0, \underline{\eta}').$$

as the unique irreducible subrepresentation. Therefore, $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}')$. This finishes the first step.

Step Two: We want to give some restrictions on $(\underline{l}', \underline{\eta}')$ in terms of $(\underline{l}, \underline{\eta})$, when $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') \neq 0$. Since

$$\underline{l}'(\cdot) = \underline{l}(\cdot) \text{ and } \underline{\eta}'(\cdot) = \underline{\eta}(\cdot)$$

over $Jord(\psi_-)$. By consideration of cuspidal supports, we are necessarily in one of the following situations.

- (1) If $\eta_k \neq (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and $\eta'_{k-1} = (-1)^{A_k-B_k}\eta'_k$, then one of the following cases is satisfied.

$$(a) \quad \eta_{k-1} = (-1)^{A_k-B_k}\eta'_{k-1} \text{ and}$$

$$(A_{k-1} - B_{k-1} - 2l_{k-1}) + (A_k - B_k - 2l_k) + 2 = (A_k - B_k - 2l'_k) - (A_{k-1} - B_{k-1} - 2l'_{k-1})$$

i.e.,

$$(l_k + l_{k-1}) - (l'_k - l'_{k-1}) = A_{k-1} - B_{k-1} + 1.$$

$$(b) \quad \eta_{k-1} \neq (-1)^{A_k-B_k}\eta'_{k-1} \text{ and}$$

$$(A_{k-1} - B_{k-1} - 2l_{k-1}) + (A_k - B_k - 2l_k) + 2 = (A_{k-1} - B_{k-1} - 2l'_{k-1}) - (A_k - B_k - 2l'_k)$$

i.e.,

$$(l_k + l_{k-1}) + (l'_k - l'_{k-1}) = A_k - B_k + 1.$$

- (2) If $\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and $\eta'_{k-1} \neq (-1)^{A_k-B_k}\eta'_k$, then one of the following cases is satisfied.

(a) $\eta_{k-1} = (-1)^{A_k - B_k} \eta'_{k-1}$ and

$$(A_k - B_k - 2l_k) - (A_{k-1} - B_{k-1} - 2l_{k-1}) = (A_{k-1} - B_{k-1} - 2l'_{k-1}) + (A_k - B_k - 2l'_k) + 2$$

i.e.,

$$(l'_k + l'_{k-1}) - (l_k - l_{k-1}) = A_{k-1} - B_{k-1} + 1.$$

(b) $\eta_{k-1} \neq (-1)^{A_k - B_k} \eta'_{k-1}$ and

$$(A_{k-1} - B_{k-1} - 2l_{k-1}) - (A_k - B_k - 2l_k) = (A_{k-1} - B_{k-1} - 2l'_{k-1}) + (A_k - B_k - 2l'_k) + 2$$

i.e.,

$$(l'_k + l'_{k-1}) + (l_k - l_{k-1}) = A_k - B_k + 1.$$

(3) If $\eta_k = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1}$ and $\eta'_{k-1} = (-1)^{A_k - B_k} \eta'_k$, then one of the following cases is satisfied.

(a) $\eta_{k-1} = (-1)^{A_k - B_k} \eta'_{k-1}$ and

$$(A_{k-1} - B_{k-1} - 2l_{k-1}) - (A_k - B_k - 2l_k) = (A_k - B_k - 2l'_k) - (A_{k-1} - B_{k-1} - 2l'_{k-1})$$

i.e.,

$$(l_k - l_{k-1}) + (l'_k - l'_{k-1}) = (A_k - B_k) - (A_{k-1} - B_{k-1}).$$

(b) $\eta_{k-1} = (-1)^{A_k - B_k} \eta'_{k-1}$ and

$$(A_k - B_k - 2l_k) - (A_{k-1} - B_{k-1} - 2l_{k-1}) = (A_{k-1} - B_{k-1} - 2l'_{k-1}) - (A_k - B_k - 2l'_k)$$

i.e.,

$$(l_k - l_{k-1}) + (l'_k - l'_{k-1}) = (A_k - B_k) - (A_{k-1} - B_{k-1}).$$

(c) $\eta_{k-1} \neq (-1)^{A_k - B_k} \eta'_{k-1}$ and

$$(A_{k-1} - B_{k-1} - 2l_{k-1}) - (A_k - B_k - 2l_k) = (A_{k-1} - B_{k-1} - 2l'_{k-1}) - (A_k - B_k - 2l'_k)$$

i.e.,

$$l_k - l_{k-1} = l'_k - l'_{k-1}.$$

(d) $\eta_{k-1} \neq (-1)^{A_k - B_k} \eta'_{k-1}$ and

$$(A_k - B_k - 2l_k) - (A_{k-1} - B_{k-1} - 2l_{k-1}) = (A_k - B_k - 2l'_k) - (A_{k-1} - B_{k-1} - 2l'_{k-1})$$

i.e.,

$$l_k - l_{k-1} = l'_k - l'_{k-1}.$$

(4) If $\eta_k \neq (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1}$ and $\eta'_{k-1} \neq (-1)^{A_k - B_k} \eta'_k$, then one of the following cases is satisfied.

(a) $\eta_{k-1} \neq (-1)^{A_k-B_k} \eta'_{k-1}$ and

$$(A_{k-1} - B_{k-1} - 2l_{k-1}) + (A_k - B_k - 2l_k) = (A_{k-1} - B_{k-1} - 2l'_{k-1}) + (A_k - B_k - 2l'_k)$$

i.e.,

$$l_k + l_{k-1} = l'_k + l'_{k-1}.$$

Since in our change of order formulas, we always have

$$\eta_{k-1} = (-1)^{A_k-B_k} \eta'_{k-1},$$

so we would like to eliminate those cases that it is not satisfied. This is not easy in general, but at least we can do this when $l_{k-1} = 0$.

Step Three: We would like to derive the change of order formula when $l_{k-1} = 0$. Let us define ψ_e by

$$Jord(\psi_e) := \cup_{C_{k-1} \in [A_{k-1}, B_{k-1}]} \{(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1})\} \cup Jord(\psi) \setminus \{(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})\}.$$

Then we can assume $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, >_\psi}^{\Sigma_0}(\psi_e, \underline{l}_e, \underline{\eta}_e) \neq 0$. Suppose

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi_e, \underline{l}_e, \underline{\eta}_e) = \pi_{M, >'_\psi}^{\Sigma_0}(\psi_e, \underline{l}'_e, \underline{\eta}'_e).$$

One can show as in **Step one** that

$$\underline{l}'_e(\cdot) = \underline{l}_e(\cdot) \text{ and } \underline{\eta}'_e(\cdot) = \underline{\eta}_e(\cdot)$$

over $Jord(\psi_-)$. Moreover, by computing $\varepsilon_{\psi_e}^{M/W}$ with respect to $>_\psi$ and $>'_\psi$, one finds $\underline{\eta}'_e$ is alternating over $\{\cup_{C_{k-1} \in [A_{k-1}, B_{k-1}]} (\rho, C_{k-1}, C_{k-1}, \zeta_{k-1})\}$. So from $(\underline{l}'_e, \underline{\eta}'_e)$, we can obtain $(\underline{l}', \underline{\eta}')$ by letting $l'_{k-1} = 0$ and $\eta'_{k-1} = \underline{\eta}'_e(\rho, B_{k-1}, B_{k-1}, \zeta)$. Then

$$\pi_{M, >'_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') = \pi_{M, >'_\psi}^{\Sigma_0}(\psi_e, \underline{l}'_e, \underline{\eta}'_e).$$

It follows from **Step two** that we have several restrictions on $(\underline{l}', \underline{\eta}')$. To eliminate the case that $\eta_{k-1} \neq (-1)^{A_k-B_k} \eta'_{k-1}$, we would like to compute the difference between η_{k-1} and η'_{k-1} explicitly. The idea is again to compute $\varepsilon_{\psi_e}^{M/W}$ with respect to $>_\psi$ and $>'_\psi$. To distinguish this two orders, we write $\varepsilon_{\psi_e}^{M/W}$ for $>_\psi$ and $\varepsilon_{\psi_e}'^{M/W}$ for $>'_\psi$. Then

$$\eta_{k-1} \varepsilon_{\psi_e}^{M/W}(\rho, B_{k-1}, B_{k-1}, \zeta) = \eta'_{k-1} \varepsilon_{\psi_e}'^{M/W}(\rho, B_{k-1}, B_{k-1}, \zeta)$$

To apply the formula for $\varepsilon_{\psi_e}^{M/W}$ (resp. $\varepsilon_{\psi_e}'^{M/W}$), we need to write $(\rho, A_k, B_k, \zeta) = (\rho, a_k, b_k)$.

- Suppose $\zeta = +1$.

$$(1) \ A_k \in \mathbb{Z}, \text{ then } \begin{cases} a_k, b_k \text{ even} & \Rightarrow \eta_{k-1} = -\eta'_{k-1} \\ a_k, b_k \text{ odd} & \Rightarrow \eta_{k-1} = \eta'_{k-1}. \end{cases}$$

$$(2) \ A_k \notin \mathbb{Z}, \text{ then } \begin{cases} a_k \text{ odd}, b_k \text{ even} & \Rightarrow \eta_{k-1} = -\eta'_{k-1} \\ a_k \text{ even}, b_k \text{ odd} & \Rightarrow \eta_{k-1} = \eta'_{k-1}. \end{cases}$$

- Suppose $\zeta = -1$.

$$(1) \ A_k \in \mathbb{Z}, \text{ then } \begin{cases} a_k, b_k \text{ even} & \Rightarrow \eta_{k-1} = -\eta'_{k-1} \\ a_k, b_k \text{ odd} & \Rightarrow \eta_{k-1} = \eta'_{k-1}. \end{cases}$$

$$(2) \ A_k \notin \mathbb{Z}, \text{ then } \begin{cases} a_k \text{ even}, b_k \text{ odd} & \Rightarrow \eta_{k-1} = -\eta'_{k-1} \\ a_k \text{ odd}, b_k \text{ even} & \Rightarrow \eta_{k-1} = \eta'_{k-1}. \end{cases}$$

It follows from the computations here that

$$\eta_{k-1} = (-1)^{\inf(a_k, b_k)-1} \eta'_{k-1}.$$

Recall $A_k - B_k + 1 = \inf(a_k, b_k)$, so this is exactly what we want. Adding this condition, the remaining cases in **Step two** are as follows.

- (1) If $\eta_k \neq (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and $\eta'_{k-1} = (-1)^{A_k-B_k}\eta'_k$, then $\eta_{k-1} = (-1)^{A_k-B_k}\eta'_{k-1}$ and $l_k - l'_k = A_{k-1} - B_{k-1} + 1$.
- (2) If $\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and $\eta'_{k-1} \neq (-1)^{A_k-B_k}\eta'_k$, then $\eta_{k-1} = (-1)^{A_k-B_k}\eta'_{k-1}$ and $l'_k - l_k = A_{k-1} - B_{k-1} + 1$.
- (3) If $\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and $\eta'_{k-1} = (-1)^{A_k-B_k}\eta'_k$, then $\eta_{k-1} = (-1)^{A_k-B_k}\eta'_{k-1}$ and $l_k + l'_k = (A_k - B_k) - (A_{k-1} - B_{k-1})$.

So this finishes the proof of the change of order formula in the case $l_{k-1} = 0$.

Step Four: In this last step, we want to show if $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') \neq 0$ for $l_{k-1} \neq 0$, then $(\underline{l}', \underline{\eta}') \sim_{\Sigma_0} S^+(\underline{l}, \underline{\eta})$. Note when $[A_k, B_k] = [A_{k-1}, B_{k-1}]$, this is obvious. So from now on, we will assume

$$[A_k, B_k] \neq [A_{k-1}, B_{k-1}].$$

First we would like to reduce it to the case $B_k = B_{k-1}$. Suppose $B_{k-1} > B_k$, let us define ψ^* from ψ by shifting $(\rho, A_{k-1}, B_{k-1}, \zeta)$ to $(\rho, A_{k-1} - B_{k-1} + B_k, B_k, \zeta)$. Then we have

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi^*, \underline{l}', \underline{\eta}') = \text{Jac}_{(\rho, A_{k-1}, B_{k-1}, \zeta) \mapsto (\rho, A_{k-1} - B_{k-1} + B_k, B_k, \zeta)} \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}').$$

So $\text{Jac}_{(\rho, A_{k-1}, B_{k-1}, \zeta) \mapsto (\rho, A_{k-1} - B_{k-1} + B_k, B_k, \zeta)}$ induces a bijection from $\Pi_{\psi}^{\Sigma_0}$ to $\Pi_{\psi^*}^{\Sigma_0}$ by Lemma 5.2. On the other side, we claim

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi^*, \underline{l}, \underline{\eta}) = \text{Jac}_{(\rho, A_{k-1}, B_{k-1}, \zeta) \mapsto (\rho, A_{k-1} - B_{k-1} + B_k, B_k, \zeta)} \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}).$$

To see this, we let ψ_{\gg} be a dominating parameter with respect to $>_\psi$, obtained from ψ by shifting (ρ, A_k, B_k, ζ) to $(\rho, A_k + T, B_k + T, \zeta)$, and ψ_{\gg} has discrete diagonal restriction. Then

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta(B_k + T) & \cdots & \zeta(B_k + 1) \\ \vdots & & \vdots \\ \zeta(A_k + T) & \cdots & \zeta(A_k + 1) \end{pmatrix} \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \\ &\hookrightarrow \underbrace{\begin{pmatrix} \zeta(B_k + T) & \cdots & \zeta(B_k + 1) \\ \vdots & & \vdots \\ \zeta(A_k + T) & \cdots & \zeta(A_k + 1) \end{pmatrix}}_{*-1} \times \underbrace{\begin{pmatrix} \zeta B_{k-1} & \cdots & \zeta(B_k + 1) \\ \vdots & & \vdots \\ \zeta A_{k-1} & \cdots & \zeta(A_{k-1} - B_{k-1} + B_k + 1) \end{pmatrix}}_{*-2} \rtimes \sigma \end{aligned}$$

where

$$\sigma := \text{Jac}_{(\rho, A_{k-1}, B_{k-1}, \zeta) \mapsto (\rho, A_{k-1} - B_{k-1} + B_k, B_k, \zeta)} \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}).$$

Since we can interchange $(*-1)$ and $(*-2)$, it is easy to see $\pi_{M, >_\psi}^{\Sigma_0}(\psi^*, \underline{l}, \underline{\eta}) = \sigma$. This shows our claim.

So if $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') \neq 0$, then $\pi_{M, >_\psi}^{\Sigma_0}(\psi^*, \underline{l}, \underline{\eta}) = \pi_{M, >_\psi}^{\Sigma_0}(\psi^*, \underline{l}', \underline{\eta}')$. And suppose we know $(\underline{l}, \underline{\eta})$ is related to $(\underline{l}', \underline{\eta}')$ according to our formula with respect to ψ^* modulo the equivalence relation \sim_{Σ_0} , then it is easy to see they are related in the same way with respect to ψ . Hence $(\underline{l}', \underline{\eta}') \sim_{\Sigma_0} S^+(\underline{l}, \underline{\eta})$.

Now we will only consider the case $B_{k-1} = B_k$, and by our previous assumption we have $A_k > A_{k-1}$. Let ψ^{**} be defined from ψ by changing (ρ, A_k, B_k, ζ) and $(\rho, A_{k-1}, B_{k-1}, \zeta)$ to $(\rho, A_k - 1, B_k + 1, \zeta)$ and $(\rho, A_{k-1} - 1, B_{k-1} + 1, \zeta)$ respectively. Then we claim for $l_{k-1} \neq 0$,

$$\begin{aligned} \text{Jac}_{\zeta B_{k-1}, \dots, -\zeta A_{k-1}} \circ \text{Jac}_{\zeta B_k, \dots, -\zeta A_k} \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) &= \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; \\ &(\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l_{k-1} - 1, \eta_{k-1}, \zeta)). \end{aligned}$$

In particular, this means we get a bijection from $\Pi_{\psi}^{\Sigma_0} \setminus \{\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) : l_{k-1} = 0\}$ to $\Pi_{\psi^{**}}^{\Sigma_0}$. To prove the claim, we will first show for $l_{k-1} \neq 0$,

$$\text{Jac}_{\zeta B_k, \dots, -\zeta A_k} \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-;$$

$$(\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta)).$$

Again let ψ_{\gg} be a dominating parameter with respect to $>_{\psi}$, obtained from ψ by shifting (ρ, A_k, B_k, ζ) to $(\rho, A_k + T, B_k + T, \zeta)$, and ψ_{\gg} has discrete diagonal restriction. Then

$$\begin{aligned} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \langle \zeta(B_k + T), \dots, -\zeta(A_k + T) \rangle \times \begin{pmatrix} \zeta(B_k + 1 + T) & \cdots & \zeta(B_k + 2) \\ \vdots & & \vdots \\ \zeta(A_k - 1 + T) & \cdots & \zeta A_k \end{pmatrix} \\ &\times \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; (\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta)) \\ &\hookrightarrow \langle \zeta(B_k + T), \dots, -\zeta A_k \rangle \times \begin{pmatrix} \zeta(B_k + 1 + T) & \cdots & \zeta(B_k + 2) \\ \vdots & & \vdots \\ \zeta(A_k - 1 + T) & \cdots & \zeta A_k \end{pmatrix} \\ &\times \underbrace{\langle -\zeta(A_k + 1), \dots, -\zeta(A_k + T) \rangle}_{** - 1} \times \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; \\ &(\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta)) \end{aligned}$$

Since $A_k > A_{k-1}$, we can take the dual of $(** - 1)$ by Corollary 3.7. Therefore,

$$\begin{aligned} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) &\hookrightarrow \langle \zeta B_k, \dots, -\zeta A_k \rangle \times \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; \\ &(\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta)) \end{aligned}$$

By applying $\text{Jac}_{\zeta B_k, \dots, -\zeta A_k}$ to the full induced representation above, we get $\pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; (\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta))$. So

$$\begin{aligned} \text{Jac}_{\zeta B_k, \dots, -\zeta A_k} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) &= \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; \\ &(\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta)). \end{aligned}$$

Next for the same T ,

$$\begin{aligned} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; (\rho, A_k - 1 + T, B_k + 1 + T, l_k - 1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta)) \\ \hookrightarrow \underbrace{\langle \zeta B_{k-1}, \dots, -\zeta A_{k-1} \rangle}_{** - 2} \times \underbrace{\begin{pmatrix} \zeta(B_k + 1 + T) & \cdots & \zeta(B_k + 2) \\ \vdots & & \vdots \\ \zeta(A_k - 1 + T) & \cdots & \zeta A_k \end{pmatrix}}_{** - 3} \\ \times \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; (\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l_{k-1} - 1, \eta_{k-1}, \zeta)). \end{aligned}$$

Since $B_k = B_{k-1}$, we can interchange $(** - 2)$ and $(** - 3)$. So

$$\begin{aligned} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; (\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta)) \\ \hookrightarrow \underbrace{\langle \zeta B_{k-1}, \dots, -\zeta A_{k-1} \rangle}_{** - 2} \times \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; (\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), \\ (\rho, A_{k-1} - 1, B_{k-1} + 1, l_{k-1} - 1, \eta_{k-1}, \zeta)). \end{aligned}$$

After applying $\text{Jac}_{\zeta B_{k-1}, \dots, -\zeta A_{k-1}}$ to the full induced representation above, we get $\pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l_{k-1} - 1, \eta_{k-1}, \zeta))$. So

$$\begin{aligned} & \text{Jac}_{\zeta B_{k-1}, \dots, -\zeta A_{k-1}} \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta)) \\ &= \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l_{k-1} - 1, \eta_{k-1}, \zeta)). \end{aligned}$$

This finishes the proof of our claim. At last, we want to compute

$$\sigma^{**} := \text{Jac}_{\zeta B_{k-1}, \dots, -\zeta A_{k-1}} \circ \text{Jac}_{\zeta B_k, \dots, -\zeta A_k} \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}')$$

for $l'_{k-1} \neq 0$. Let ψ'_{\gg} be a dominating parameter with respect to $>'_\psi$, obtained from ψ by shifting $(\rho, A_{k-1}, B_{k-1}, \zeta)$ to $(\rho, A_{k-1} + T', B_{k-1} + T', \zeta)$, and ψ'_{\gg} has discrete diagonal restriction. Then

$$\begin{aligned} \pi_{M, >'_\psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') &\hookrightarrow \underbrace{\langle \zeta(B_{k-1} + T'), \dots, -\zeta(A_{k-1} + T') \rangle}_I \times \underbrace{\langle \zeta B_k, \dots, -\zeta A_k \rangle}_{III} \\ &\times \begin{pmatrix} \zeta(B_{k-1} + 1 + T') & \cdots & \zeta(B_{k-1} + 2) \\ \vdots & & \vdots \\ \zeta(A_{k-1} - 1 + T') & \cdots & \zeta A_{k-1} \end{pmatrix} \rtimes \pi_{M, >'_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; \\ &(\rho, A_k - 1, B_k + 1, l'_k - 1, \eta'_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta)) \\ &\hookrightarrow \underbrace{\langle \zeta(B_{k-1} + T'), \dots, \zeta(B_{k-1} + 1) \rangle}_I \times \underbrace{\langle \zeta B_{k-1}, \dots, -\zeta(A_{k-1} + T') \rangle}_{II} \\ &\times \underbrace{\langle \zeta B_k, \dots, -\zeta A_k \rangle}_{III} \times \underbrace{\begin{pmatrix} \zeta(B_{k-1} + 1 + T') & \cdots & \zeta(B_{k-1} + 2) \\ \vdots & & \vdots \\ \zeta(A_{k-1} - 1 + T') & \cdots & \zeta A_{k-1} \end{pmatrix}}_{IV} \\ &\rtimes \pi_{M, >'_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k - 1, B_k + 1, l'_k - 1, \eta'_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta)). \end{aligned}$$

We can interchange (IV) with (III) and (II). Also (II) and (III) are interchangeable. So

$$\begin{aligned} \pi_{M, >'_\psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') &\hookrightarrow \underbrace{\langle \zeta(B_{k-1} + T'), \dots, \zeta(B_{k-1} + 1) \rangle}_I \times \underbrace{\begin{pmatrix} \zeta(B_{k-1} + 1 + T') & \cdots & \zeta(B_{k-1} + 2) \\ \vdots & & \vdots \\ \zeta(A_{k-1} - 1 + T') & \cdots & \zeta A_{k-1} \end{pmatrix}}_{IV} \\ &\times \underbrace{\langle \zeta B_k, \dots, -\zeta A_k \rangle}_{III} \times \underbrace{\langle \zeta B_{k-1}, \dots, -\zeta(A_{k-1} + T') \rangle}_{II} \\ &\rtimes \pi_{M, >'_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k - 1, B_k + 1, l'_k - 1, \eta'_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta)) \\ &\hookrightarrow \underbrace{\langle \zeta(B_{k-1} + T'), \dots, \zeta(B_{k-1} + 1) \rangle}_I \times \underbrace{\begin{pmatrix} \zeta(B_{k-1} + 1 + T') & \cdots & \zeta(B_{k-1} + 2) \\ \vdots & & \vdots \\ \zeta(A_{k-1} - 1 + T') & \cdots & \zeta A_{k-1} \end{pmatrix}}_{IV} \\ &\times \underbrace{\langle \zeta B_k, \dots, -\zeta A_{k-1} \rangle}_{III_1} \times \underbrace{\langle -\zeta(A_{k-1} + 1), \dots, -\zeta A_k \rangle}_{III_2} \times \underbrace{\langle \zeta B_{k-1}, \dots, -\zeta(A_{k-1} + T') \rangle}_{II} \end{aligned}$$

$$\times \underbrace{\begin{pmatrix} \zeta(B_k + 1) & \cdots & \zeta(B_k + A_{k-1} - A_k + 2) \\ \vdots & & \vdots \\ \zeta(A_k - 1) & \cdots & \zeta A_{k-1} \end{pmatrix}}_V \rtimes \pi_{M, >'_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; \\ (\rho, A_{k-1} - 1, B_k + A_{k-1} - A_k + 1, l'_k - 1, \eta'_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta))$$

Since $\text{Jac}_{\zeta(B_{k-1}+1+T')} \pi_{M, >'_\psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') = 0$, we can combine (I) and (IV). We can also interchange (III₂) with (II) and (V), and then take dual of (III₂). As a result,

$$\pi_{M, >'_\psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B_{k-1} + T') & \cdots & \zeta(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta(A_{k-1} - 1 + T') & \cdots & \zeta A_{k-1} \end{pmatrix}}_{IV_+} \times \underbrace{\langle \zeta B_k, \dots, -\zeta A_{k-1} \rangle}_{III_1} \\ \times \underbrace{\langle \zeta B_{k-1}, \dots, -\zeta(A_{k-1} + T') \rangle}_{II} \times \underbrace{\begin{pmatrix} \zeta(B_k + 1) & \cdots & \zeta(B_k + A_{k-1} - A_k + 2) \\ \vdots & & \vdots \\ \zeta(A_k - 1) & \cdots & \zeta A_{k-1} \end{pmatrix}}_V \\ \times \underbrace{\langle \zeta A_k, \dots, \zeta(A_{k-1} + 1) \rangle}_{(III_2)^\vee} \rtimes \pi_{M, >'_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; \\ (\rho, A_{k-1} - 1, B_k + A_{k-1} - A_k + 1, l'_k - 1, \eta'_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta))$$

Since $A_k > A_{k-1} > B_{k-1}$ and $\text{Jac}_{\zeta A_k} \pi_{M, >'_\psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') = 0$, we can further combine (III₂)[∨] and (V).

$$\pi_{M, >'_\psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B_{k-1} + T') & \cdots & \zeta(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta(A_{k-1} - 1 + T') & \cdots & \zeta A_{k-1} \end{pmatrix}}_{IV_+} \times \underbrace{\langle \zeta B_k, \dots, -\zeta A_{k-1} \rangle}_{III_1} \\ \times \underbrace{\langle \zeta B_{k-1}, \dots, -\zeta(A_{k-1} + T') \rangle}_{II} \times \underbrace{\begin{pmatrix} \zeta(B_k + 1) & \cdots & \zeta(B_k + A_{k-1} - A_k + 2) \\ \vdots & & \vdots \\ \zeta A_k & \cdots & \zeta(A_{k-1} + 1) \end{pmatrix}}_{V_+} \\ \rtimes \pi_{M, >'_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_{k-1} - 1, B_k + A_{k-1} - A_k + 1, l'_k - 1, \eta'_k, \zeta), \\ (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta)) \\ \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B_{k-1} + T') & \cdots & \zeta(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta(A_{k-1} - 1 + T') & \cdots & \zeta A_{k-1} \end{pmatrix}}_{IV_+} \times \underbrace{\langle \zeta B_k, \dots, -\zeta A_{k-1} \rangle}_{III_1} \\ \times \underbrace{\langle \zeta B_{k-1}, \dots, -\zeta A_{k-1} \rangle}_{II_1} \times \underbrace{\langle -\zeta(A_{k-1} + 1), \dots, -\zeta(A_{k-1} + T') \rangle}_{II_2}$$

$$\times \underbrace{\begin{pmatrix} \zeta(B_k + 1) & \cdots & \zeta(B_k + A_{k-1} - A_k + 2) \\ \vdots & & \vdots \\ \zeta A_k & \cdots & \zeta(A_{k-1} + 1) \end{pmatrix}}_{V_+} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; \\ (\rho, A_{k-1} - 1, B_k + A_{k-1} - A_k + 1, l'_k - 1, \eta'_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta))$$

So we can interchange (II_2) with (V_+) , and take dual of (II_2) . Note $(II_2)^\vee$ is interchangeable with (V_+) . Since $A_{k-1} > B_{k-1}$, $(II_2)^\vee$ is also interchangeable with (II_1) and (III_1) . Therefore,

$$\pi_{M, > \psi}^{\Sigma_0}(\psi'_{\gg}, \underline{l}', \underline{\eta}') \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B_{k-1} + T') & \cdots & \zeta(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta(A_{k-1} - 1 + T') & \cdots & \zeta A_{k-1} \end{pmatrix}}_{IV_+} \times \underbrace{\langle \zeta(A_{k-1} + T'), \dots, \zeta(A_{k-1} + 1) \rangle}_{(II_2)^\vee} \\ \times \underbrace{\langle \zeta B_k, \dots, -\zeta A_{k-1} \rangle}_{III_1} \times \underbrace{\langle \zeta B_{k-1}, \dots, -\zeta A_{k-1} \rangle}_{II_1} \\ \times \underbrace{\begin{pmatrix} \zeta(B_k + 1) & \cdots & \zeta(B_k + A_{k-1} - A_k + 2) \\ \vdots & & \vdots \\ \zeta A_k & \cdots & \zeta(A_{k-1} + 1) \end{pmatrix}}_{V_+} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; \\ (\rho, A_{k-1} - 1, B_k + A_{k-1} - A_k + 1, l'_k - 1, \eta'_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta)).$$

Consequently,

$$\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') \hookrightarrow \underbrace{\langle \zeta B_k, \dots, -\zeta A_{k-1} \rangle}_{III_1} \times \underbrace{\langle \zeta B_{k-1}, \dots, -\zeta A_{k-1} \rangle}_{II_1} \\ \times \underbrace{\begin{pmatrix} \zeta(B_k + 1) & \cdots & \zeta(B_k + A_{k-1} - A_k + 2) \\ \vdots & & \vdots \\ \zeta A_k & \cdots & \zeta(A_{k-1} + 1) \end{pmatrix}}_{V_+} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; \\ (\rho, A_{k-1} - 1, B_k + A_{k-1} - A_k + 1, l'_k - 1, \eta'_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta)) \\ \hookrightarrow \underbrace{\langle \zeta B_k, \dots, -\zeta A_{k-1} \rangle}_{III_1} \times \underbrace{\langle \zeta B_{k-1}, \dots, -\zeta A_{k-1} \rangle}_{II_1} \\ \times \underbrace{\begin{pmatrix} \zeta(B_k + 1) & \cdots & \zeta(B_k + A_{k-1} - A_k + 2) \\ \vdots & & \vdots \\ \zeta(A_k - 1) & \cdots & \zeta A_{k-1} \end{pmatrix}}_V \times \underbrace{\langle \zeta A_k, \dots, \zeta(A_{k-1} + 1) \rangle}_{(III_2)^\vee} \\ \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_{k-1} - 1, B_k + A_{k-1} - A_k + 1, l'_k - 1, \eta'_k, \zeta), \\ (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta)).$$

Then we take dual of $(III_2)^\vee$, and interchange (III_2) with (V) .

$$\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') \hookrightarrow \underbrace{\langle \zeta B_k, \dots, -\zeta A_{k-1} \rangle}_{III_1} \times \underbrace{\langle \zeta B_{k-1}, \dots, -\zeta A_{k-1} \rangle}_{II_1}$$

$$\begin{aligned}
& \underbrace{< -\zeta(A_{k-1} + 1), \dots, -\zeta A_k >}_{III_2} \times \underbrace{\begin{pmatrix} \zeta(B_k + 1) & \cdots & \zeta(B_k + A_{k-1} - A_k + 2) \\ \vdots & & \vdots \\ \zeta(A_k - 1) & \cdots & \zeta A_{k-1} \end{pmatrix}}_V \\
& \rtimes \pi_{M, >'_\psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_{k-1} - 1, B_k + A_{k-1} - A_k + 1, l'_k - 1, \eta'_k, \zeta), \right. \\
& \left. (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta) \right).
\end{aligned}$$

Suppose

$$\begin{aligned}
\pi_{M, >'_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') & \hookrightarrow \underbrace{< \zeta B_k, \dots, -\zeta A_{k-1} >}_{III_1} \times \underbrace{< \zeta B_{k-1}, \dots, -\zeta A_k >}_{II_1 + III_2} \\
& \times \underbrace{\begin{pmatrix} \zeta(B_k + 1) & \cdots & \zeta(B_k + A_{k-1} - A_k + 2) \\ \vdots & & \vdots \\ \zeta(A_k - 1) & \cdots & \zeta A_{k-1} \end{pmatrix}}_V \rtimes \pi_{M, >'_\psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; \right. \\
& \left. (\rho, A_{k-1} - 1, B_k + A_{k-1} - A_k + 1, l'_k - 1, \eta'_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta) \right).
\end{aligned}$$

Since we can interchange (III_1) with $(II_1 + III_2)$, and $B_k = B_{k-1}$, we have

$$\begin{aligned}
\sigma^{**} & \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B_k + 1) & \cdots & \zeta(B_k + A_{k-1} - A_k + 2) \\ \vdots & & \vdots \\ \zeta(A_k - 1) & \cdots & \zeta A_{k-1} \end{pmatrix}}_V \rtimes \pi_{M, >'_\psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; \right. \\
& \left. (\rho, A_{k-1} - 1, B_k + A_{k-1} - A_k + 1, l'_k - 1, \eta'_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta) \right).
\end{aligned}$$

Otherwise, we would have

$$\begin{aligned}
\pi_{M, >'_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') & \hookrightarrow \underbrace{< \zeta B_k, \dots, -\zeta A_{k-1} >}_{III_1} \times \underbrace{< -\zeta(A_{k-1} + 1), \dots, -\zeta A_k >}_{III_2} \\
& \underbrace{< \zeta B_{k-1}, \dots, -\zeta A_{k-1} >}_{II_1} \times \underbrace{\begin{pmatrix} \zeta(B_k + 1) & \cdots & \zeta(B_k + A_{k-1} - A_k + 2) \\ \vdots & & \vdots \\ \zeta(A_k - 1) & \cdots & \zeta A_{k-1} \end{pmatrix}}_V \\
& \rtimes \pi_{M, >'_\psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_{k-1} - 1, B_k + A_{k-1} - A_k + 1, l'_k - 1, \eta'_k, \zeta), \right. \\
& \left. (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta) \right).
\end{aligned}$$

Then we again have

$$\begin{aligned}
\sigma^{**} & \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B_k + 1) & \cdots & \zeta(B_k + A_{k-1} - A_k + 2) \\ \vdots & & \vdots \\ \zeta(A_k - 1) & \cdots & \zeta A_{k-1} \end{pmatrix}}_V \rtimes \pi_{M, >'_\psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; \right. \\
& \left. (\rho, A_{k-1} - 1, B_k + A_{k-1} - A_k + 1, l'_k - 1, \eta'_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta) \right).
\end{aligned}$$

Note the full induced representation above has a unique irreducible subrepresentation:

$$\pi_{M, >'_\psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k - 1, B_k + 1, l'_k - 1, \eta'_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta) \right).$$

So it must be equal to σ^{**} . To summarize, if $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, > \psi'}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') \neq 0$ for $l_{k-1} \neq 0$, then we have $l'_{k-1} \neq 0$ by the previous step. After applying

$$\text{Jac}_{\zeta B_{k-1}, \dots, -\zeta A_{k-1}} \circ \text{Jac}_{\zeta B_k, \dots, -\zeta A_k}$$

to both sides we get

$$\begin{aligned} & \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}; (\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l_{k-1} - 1, \eta_{k-1}, \zeta)) \\ &= \pi_{M, > \psi'}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}; (\rho, A_k - 1, B_k + 1, l'_k - 1, \eta'_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta)). \end{aligned}$$

By induction on l_{k-1} , we can assume $(l_k - 1, \eta_k; l_{k-1} - 1, \eta_{k-1})$ is related to $(l'_k - 1, \eta'_k; l'_{k-1} - 1, \eta'_{k-1})$ according to our formula with respect to ψ^{**} . Then it is easy to deduce that $(l_k, \eta_k; l_{k-1}, \eta_{k-1})$ and $(l'_k, \eta'_k; l'_{k-1}, \eta'_{k-1})$ are also related according to our formula with respect to ψ . Hence $(\underline{l}', \underline{\eta}') \sim_{\Sigma_0} S^+(\underline{l}, \underline{\eta})$.

5.2. Case $\zeta_k \neq \zeta_{k-1}$. In this case, there is no extra conditions on $[A_k, B_k], [A_{k-1}, B_{k-1}]$. For functions $\underline{l}(\rho, A, B, \zeta) \in [0, [(A - B + 1)/2]]$ and $\underline{\eta}(\rho, A, B, \zeta) \in \mathbb{Z}_2$ on $Jord(\psi)$, we denote

$$l_k = \underline{l}(\rho, A_k, B_k, \zeta_k), \quad l_{k-1} = \underline{l}(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}),$$

and

$$\eta_k = \underline{\eta}(\rho, A_k, B_k, \zeta_k), \quad \eta_{k-1} = \underline{\eta}(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}).$$

From $(\underline{l}, \underline{\eta})$, we want to construct another pair $(\underline{l}', \underline{\eta}')$ such that

$$\underline{l}'(\cdot) = \underline{l}(\cdot) \text{ and } \underline{\eta}'(\cdot) = \underline{\eta}(\cdot)$$

over $Jord(\psi) \setminus \{(\rho, A_k, B_k, \zeta_k), (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})\}$. Let us denote

$$l'_k = \underline{l}'(\rho, A_k, B_k, \zeta_k), \quad l'_{k-1} = \underline{l}(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}),$$

and

$$\eta'_k = \underline{\eta}'(\rho, A_k, B_k, \zeta_k), \quad \eta'_{k-1} = \underline{\eta}'(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}).$$

Then we define $l'_k, l'_{k-1}, \eta'_k, \eta'_{k-1}$ according to the following formulas.

$$\begin{cases} l'_k = l_k \\ l'_{k-1} = l_{k-1} \\ \eta_k = (-1)^{A_{k-1} - B_{k-1} + 1} \eta'_k \\ \eta_{k-1} = (-1)^{A_k - B_k + 1} \eta'_{k-1} \end{cases}$$

We denote this transformation by U . Since the situation is symmetric here, we have $U \circ U = id$.

Proposition 5.3. Suppose $(\underline{l}', \underline{\eta}') = U(\underline{l}, \underline{\eta})$, then

$$\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, > \psi'}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}').$$

Let ψ_{\gg} be a dominating parameter of ψ such that $Jord_{\rho}(\psi_{\gg}) = Jord_{\rho}(\psi)$, and the Jordan blocks in $Jord_{\rho'}(\psi_{\gg})$ are disjoint for $\rho' \neq \rho$. Then

$$\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \text{Jac}_{X^c} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}),$$

and

$$\pi_{M, > \psi'}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') = \text{Jac}_{X^c} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}', \underline{\eta}').$$

So it suffices to prove the proposition for such ψ_{\gg} . Therefore, in the following discussions of the proof of this proposition, we will always assume the Jordan blocks in $Jord_{\rho'}(\psi)$ are disjoint for $\rho' \neq \rho$, and if we choose some dominating ψ_{\gg} of ψ , we will always assume $Jord_{\rho'}(\psi_{\gg}) = Jord_{\rho'}(\psi)$ for $\rho' \neq \rho$.

5.2.1. *First reduction.* Let $(\underline{l}', \underline{\eta}') = U(\underline{l}, \underline{\eta})$. We want to reduce the proposition to the following cases:

$$(5.3) \quad (\rho, A_i, B_i, \zeta_i) \gg_r (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}) \text{ for all } i, \text{ and } (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}) \gg_r 0.$$

We denote the case with respect to r by $(5.3)_r$. We will do this in two steps. First we will reduce it to the cases:

$$(5.4) \quad \begin{cases} (\rho, A_i, B_i, \zeta_i) \gg_r (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}) \text{ for } i > k-1, \\ (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}) \gg_r \cup_{j=1}^{k-2} \{(\rho, A_j, B_j, \zeta_j)\} \text{ and } 0. \end{cases}$$

We denote the case with respect to r by $(5.4)_r$. Let us choose a dominating parameter ψ_{\gg} with respect to $>_{\psi}$ such that $T_i = 0$ for $i < k-1$,

$$(\rho, A_i + T_i, B_i + T_i, \zeta_i) \gg_r (\rho, A_{i-1} + T_{i-1}, B_{i-1} + T_{i-1}, \zeta_{i-1}) \text{ for } i \geq k.$$

We further require the existence of T such that $0 \leq T < T_k$,

$$(\rho, A_{k-1} + T_{k-1}, B_{k-1} + T_{k-1}, \zeta_{k-1}) \gg_r (\rho, A_k + T_k - T, B_k + T_k - T, \zeta_k) \gg_r \cup_{j=1}^{k-2} \{(\rho, A_j, B_j, \zeta_j)\} \text{ and } 0.$$

Let $\psi_{\gg}^{(k)}$ be obtained from ψ_{\gg} by changing T_k, T_{k-1} to zero. Suppose $\pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, then

$$\begin{aligned} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix} \times \begin{pmatrix} \zeta_k(B_k + T_k) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k) & \cdots & \zeta_k(A_k + 1) \end{pmatrix} \\ &\rtimes \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}, \underline{\eta}), \end{aligned}$$

where the two generalized segments are interchangeable. Let ψ_{\gg}^T be obtained from ψ_{\gg} by changing $(\rho, A_k + T_k, B_k + T_k, \zeta_k)$ to $(\rho, A_k + T_k - T, B_k + T_k - T, \zeta_k)$. Then

$$\begin{aligned} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}^T, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta_k(B_k + T_k - T) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k - T) & \cdots & \zeta_k(A_k + 1) \end{pmatrix} \times \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix} \\ &\rtimes \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}, \underline{\eta}), \end{aligned}$$

By $(5.4)_r$, we have

$$\pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}^T, \underline{l}, \underline{\eta}) = \pi_{M, >_{\psi}'}^{\Sigma_0}(\psi_{\gg}^T, \underline{l}', \underline{\eta}').$$

Since

$$\begin{aligned} \pi_{M, >_{\psi}'}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}', \underline{\eta}') &= \text{Jac}_{(\rho, A_{k-1} + T_{k-1}, B_{k-1} + T_{k-1}, \zeta_{k-1}) \mapsto (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})} \circ \\ &\quad \text{Jac}_{(\rho, A_k + T_k - T, B_k + T_k - T, \zeta_k) \mapsto (\rho, A_k, B_k, \zeta_k)} \pi_{M, >_{\psi}'}^{\Sigma_0}(\psi_{\gg}^T, \underline{l}', \underline{\eta}') \end{aligned}$$

and $\pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}, \underline{\eta})$ is contained in

$$\begin{aligned} &\text{Jac}_{(\rho, A_{k-1} + T_{k-1}, B_{k-1} + T_{k-1}, \zeta_{k-1}) \mapsto (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})} \circ \\ &\quad \text{Jac}_{(\rho, A_k + T_k - T, B_k + T_k - T, \zeta_k) \mapsto (\rho, A_k, B_k, \zeta_k)} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}^T, \underline{l}, \underline{\eta}), \end{aligned}$$

then

$$\pi_{M, >_{\psi}}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}, \underline{\eta}) = \pi_{M, >_{\psi}'}^{\Sigma_0}(\psi_{\gg}^{(k)}, \underline{l}', \underline{\eta}').$$

After applying $\circ_{i > k} \text{Jac}_{(\rho, A_i + T_i, B_i + T_i, \zeta_i) \mapsto (\rho, A_i, B_i, \zeta_i)}$ to both sides, we get $\pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, >_{\psi}'}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}')$.

Secondly we want to further reduce it to $(5.3)_r$. Let us assume we are in case $(5.4)_{r'}$ for r' sufficiently large with respect to r . We can choose a dominating parameter ψ_{\gg} with respect to $>_{\psi}$ such that $T_i = 0$ for $i > k$, and

$$(\rho, A_{k+1}, B_{k+1}, \zeta_{k+1}) \gg_r (\rho, A_i + T_i, B_i + T_i, \zeta_i) \gg_r (\rho, A_{i-1} + T_{i-1}, B_{i-1} + T_{i-1}, \zeta_{i-1}) \text{ for } i \leq k.$$

Suppose $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, then

$$\begin{aligned} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \times_{i < k-1} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix} \times \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix} \\ &\quad \times \begin{pmatrix} \zeta_k(B_k + T_k) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k) & \cdots & \zeta_k(A_k + 1) \end{pmatrix} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}), \end{aligned}$$

where i increases. We can also assume $B_{k-1} + 1 > A_{k-2} + T_{k-2} + 1$. Then we can change the order of the generalized segments as follows,

$$\begin{aligned} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix} \times \begin{pmatrix} \zeta_k(B_k + T_k) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k) & \cdots & \zeta_k(A_k + 1) \end{pmatrix} \\ &\quad \times_{i < k-1} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \\ &\cong \begin{pmatrix} \zeta_k(B_k + T_k) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k) & \cdots & \zeta_k(A_k + 1) \end{pmatrix} \times \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix} \\ &\quad \times_{i < k-1} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}). \end{aligned}$$

We can choose $0 \leq T < T_k$ such that

$(\rho, A_{k-1} + T_{k-1}, B_{k-1} + T_{k-1}, \zeta_{k-1}) \gg_r (\rho, A_k + T_k - T, B_k + T_k - T, \zeta_k) \gg_r (\rho, A_{k-2} + T_{k-2}, B_{k-2} + T_{k-2}, \zeta_{k-2})$ and 0.

Let ψ_{\gg}^T be obtained from ψ_{\gg} by changing $(\rho, A_k + T_k, B_k + T_k, \zeta_k)$ to $(\rho, A_k + T_k - T, B_k + T_k - T, \zeta_k)$. Then

$$\begin{aligned} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}^T, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta_k(B_k + T_k - T) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k - T) & \cdots & \zeta_k(A_k + 1) \end{pmatrix} \times \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix} \\ &\quad \times_{i < k-1} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \\ &\cong \times_{i < k-1} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix} \times \begin{pmatrix} \zeta_k(B_k + T_k - T) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k - T) & \cdots & \zeta_k(A_k + 1) \end{pmatrix} \\ &\quad \times \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}). \end{aligned}$$

By (5.3)_r, we have

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}^T, \underline{l}, \underline{\eta}) = \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}^T, \underline{l}', \underline{\eta}').$$

Since

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') &= \text{Jac}_{(\rho, A_{k-1}+T_{k-1}, B_{k-1}+T_{k-1}, \zeta_{k-1}) \mapsto (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})} \circ \\ &\quad \text{Jac}_{(\rho, A_k+T_k-T, B_k+T_k-T, \zeta_k) \mapsto (\rho, A_k, B_k, \zeta_k)} \circ \\ &\quad \circ_{i < k-1} \text{Jac}_{(\rho, A_i+T_i, B_i+T_i, \zeta_i) \mapsto (\rho, A_i, B_i, \zeta_i)} \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}^T, \underline{l}', \underline{\eta}') \end{aligned}$$

and $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ is contained in

$$\begin{aligned} &\text{Jac}_{(\rho, A_{k-1}+T_{k-1}, B_{k-1}+T_{k-1}, \zeta_{k-1}) \mapsto (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})} \circ \\ &\text{Jac}_{(\rho, A_k+T_k-T, B_k+T_k-T, \zeta_k) \mapsto (\rho, A_k, B_k, \zeta_k)} \circ \\ &\circ_{i < k-1} \text{Jac}_{(\rho, A_i+T_i, B_i+T_i, \zeta_i) \mapsto (\rho, A_i, B_i, \zeta_i)} \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}^T, \underline{l}, \underline{\eta}), \end{aligned}$$

then

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}').$$

This finishes the first reduction.

5.2.2. *Second reduction.* We want to reduce the proposition further to the cases:

$$(5.5) \quad (\rho, A_i, B_i, \zeta_i) \gg_r (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}), \text{ and } l_i = 0 \text{ for all } i.$$

Let us denote the case with respect to r by $(5.5)_r$. Suppose we are in case $(5.3)_{r'}$ for r' sufficiently large with respect to r . Let ψ^T be obtained by changing $(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})$ to $(\rho, A_{k-1}+T, B_{k-1}+T, \zeta_{k-1})$ such that

$$B_{k-1}+T > A_k \text{ and } B_{k+1} > A_{k-1}+T.$$

Let

$$\text{Jord}(\psi_-) = \{(\rho, A_i - l_i, B_i + l_i, \zeta_i) : i \neq k, k-1\}.$$

We define $(\underline{l}_-, \underline{\eta}_-)$ such that $\underline{l}_-(\rho, A_i - l_i, B_i + l_i, \zeta_i) = 0$ and $\underline{\eta}_-(\rho, A_i - l_i, B_i + l_i, \zeta_i) = \eta_i$. Then

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi^T, \underline{l}', \underline{\eta}') &\hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T) & \cdots & -\zeta_{k-1}(A_{k-1}+T) \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1}+l_{k-1}-1+T) & \cdots & -\zeta_{k-1}(A_{k-1}-l_{k-1}+1+T) \end{pmatrix} \\ &\times_{i \neq k-1} \begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i + l_i - 1) & \cdots & -\zeta_i(A_i - l_i + 1) \end{pmatrix} \\ &\times \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_{k-1} - l_{k-1} + T, B_{k-1} + l_{k-1} + T, 0, \eta'_{k-1}, \zeta_{k-1}), \\ &(\rho, A_k - l_k, B_k + l_k, 0, \eta'_k, \zeta_k)). \end{aligned}$$

We choose t such that

$$(\rho, A_{k-1} - l_{k-1}, B_{k-1} + l_{k-1}, \zeta_{k-1}) \gg_r (\rho, A_k - l_k - t, B_k + l_k - t, \zeta_k) \gg_r (\rho, A_{k-2} - l_{k-2}, B_{k-2} + l_{k-2}, \zeta_{k-2}).$$

Then

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi^T, \underline{l}', \underline{\eta}') &\hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T) & \cdots & -\zeta_{k-1}(A_{k-1}+T) \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1}+l_{k-1}-1+T) & \cdots & -\zeta_{k-1}(A_{k-1}-l_{k-1}+1+T) \end{pmatrix} \\ &\times_{i \neq k-1} \underbrace{\begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i + l_i - 1) & \cdots & -\zeta_i(A_i - l_i + 1) \end{pmatrix}}_{I_i} \times \underbrace{\begin{pmatrix} \zeta_k(B_k + l_k) & \cdots & \zeta_k(B_k + l_k - t + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k - l_k) & \cdots & \zeta_k(A_k - l_k - t + 1) \end{pmatrix}}_{II} \end{aligned}$$

$$\begin{aligned}
& \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + l_{k-1} + T) & \cdots & \zeta_{k-1}(B_{k-1} + l_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} - l_{k-1} + T) & \cdots & \zeta_{k-1}(A_{k-1} - l_{k-1} + 1) \end{pmatrix}}_{III} \\
& \times \pi_{M, >_\psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1} - l_{k-1}, B_{k-1} + l_{k-1}, 0, \eta'_{k-1}, \zeta_{k-1}), \right. \\
& \left. (\rho, A_k - l_k - t, B_k + l_k - t, 0, \eta'_k, \zeta_k) \right).
\end{aligned}$$

It is clear that the generalized segments (III) and (II) are interchangeable. We would like to show (III) and (I_i) are also interchangeable for $i \neq k-1$. It suffices to make the following observations:

- (1) If $\zeta_i = \zeta_{k-1}$
 - (a) $i > k$, one observes $B_i > B_{k-1} + l_{k-1} + T$
 - (b) $i < k-1$, one observes $B_{k-1} + l_{k-1} > B_i + l_i$
- (2) If $\zeta_i \neq \zeta_{k-1}$
 - (a) $i > k$, one observes $A_i > A_{k-1} - l_{k-1} + T$
 - (b) $i < k-1$, one observes $B_{k-1} + l_{k-1} > A_i$
 - (c) $i = k$, one observes $[A_k, A_k - l_k + 1] \subseteq [A_{k-1} + T - l_{k-1}, A_{k-1} - l_{k-1} + 1]$

Therefore,

$$\begin{aligned}
\pi_{M, >_\psi}^{\Sigma_0}(\psi^T, \underline{l}', \underline{\eta}') & \hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T) & \cdots & -\zeta_{k-1}(A_{k-1} + T) \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1} + l_{k-1} - 1 + T) & \cdots & -\zeta_{k-1}(A_{k-1} - l_{k-1} + 1 + T) \end{pmatrix} \\
& \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + l_{k-1} + T) & \cdots & \zeta_{k-1}(B_{k-1} + l_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} - l_{k-1} + T) & \cdots & \zeta_{k-1}(A_{k-1} - l_{k-1} + 1) \end{pmatrix}}_{III} \\
& \times_{i \neq k-1} \underbrace{\begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i + l_i - 1) & \cdots & -\zeta_i(A_i - l_i + 1) \end{pmatrix}}_{I_i} \times \underbrace{\begin{pmatrix} \zeta_k(B_k + l_k) & \cdots & \zeta_k(B_k + l_k - t + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k - l_k) & \cdots & \zeta_k(A_k - l_k - t + 1) \end{pmatrix}}_{II} \\
& \times \pi_{M, >_\psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1} - l_{k-1}, B_{k-1} + l_{k-1}, 0, \eta'_{k-1}, \zeta_{k-1}), \right. \\
& \left. (\rho, A_k - l_k - t, B_k + l_k - t, 0, \eta'_k, \zeta_k) \right).
\end{aligned}$$

Next we want to take dual of

$$\underbrace{\begin{pmatrix} -\zeta_{k-1}(A_{k-1} + 1) & \cdots & -\zeta_{k-1}(A_{k-1} + T) \\ \vdots & & \vdots \\ -\zeta_{k-1}(A_{k-1} - l_{k-1} + 2) & \cdots & -\zeta_{k-1}(A_{k-1} - l_{k-1} + 1 + T) \end{pmatrix}}_{IV}$$

from

$$\begin{pmatrix} \zeta_{k-1}(B_{k-1} + T) & \cdots & -\zeta_{k-1}(A_{k-1} + T) \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1} + l_{k-1} - 1 + T) & \cdots & -\zeta_{k-1}(A_{k-1} - l_{k-1} + 1 + T) \end{pmatrix}.$$

It is clear that (IV) and (III) are interchangeable. To see (IV) and (II) are interchangeable, one notes $\zeta_k \neq \zeta_{k-1}$ and $[A_{k-1} + T, A_{k-1} + 1] \supseteq [A_k - l_k, B_k + l_k]$. To see (IV) and (I_i) are also interchangeable, it suffices to make the following observations:

- (1) If $\zeta_i = \zeta_{k-1}$
 - (a) $i > k$, one observes $A_i > A_{k-1} + T$
 - (b) $i < k - 1$, one observes $A_{k-1} + l_{k-1} > A_i$
- (2) If $\zeta_i \neq \zeta_{k-1}$
 - (a) $i \geq k$, one observes $B_i > A_{k-1} + 1$
 - (b) $i < k - 1$, one observes $A_{k-1} - l_{k-1} > B_i + l_i$

As a result,

$$\begin{aligned}
\pi_{M, >'_\psi}^{\Sigma_0}(\psi^T, \underline{l}', \underline{\eta}') &\hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T) & \cdots & -\zeta_{k-1}A_{k-1} \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1} + l_{k-1} - 1 + T) & \cdots & -\zeta_{k-1}(A_{k-1} - l_{k-1} + 1) \end{pmatrix} \\
&\times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + l_{k-1} + T) & \cdots & \zeta_{k-1}(B_{k-1} + l_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} - l_{k-1} + T) & \cdots & \zeta_{k-1}(A_{k-1} - l_{k-1} + 1) \end{pmatrix}}_{III} \\
&\times_{i \neq k-1} \underbrace{\begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i + l_i - 1) & \cdots & -\zeta_i(A_i - l_i + 1) \end{pmatrix}}_{I_i} \times \underbrace{\begin{pmatrix} \zeta_k(B_k + l_k) & \cdots & \zeta_k(B_k + l_k - t + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k - l_k) & \cdots & \zeta_k(A_k - l_k - t + 1) \end{pmatrix}}_{II} \\
&\times \underbrace{\begin{pmatrix} -\zeta_{k-1}(A_{k-1} + 1) & \cdots & -\zeta_{k-1}(A_{k-1} + T) \\ \vdots & & \vdots \\ -\zeta_{k-1}(A_{k-1} - l_{k-1} + 2) & \cdots & -\zeta_{k-1}(A_{k-1} - l_{k-1} + 1 + T) \end{pmatrix}}_{IV} \\
&\times \pi_{M, >'_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1} - l_{k-1}, B_{k-1} + l_{k-1}, 0, \eta'_{k-1}, \zeta_{k-1}), \\
&(\rho, A_k - l_k - t, B_k + l_k - t, 0, \eta'_k, \zeta_k)).
\end{aligned}$$

By Corollary 3.7, we can take the dual of (IV). Therefore

$$\begin{aligned}
\pi_{M, >'_\psi}^{\Sigma_0}(\psi^T, \underline{l}', \underline{\eta}') &\hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T) & \cdots & -\zeta_{k-1}A_{k-1} \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1} + l_{k-1} - 1 + T) & \cdots & -\zeta_{k-1}(A_{k-1} - l_{k-1} + 1) \end{pmatrix} \\
&\times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + l_{k-1} + T) & \cdots & \zeta_{k-1}(B_{k-1} + l_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} - l_{k-1} + T) & \cdots & \zeta_{k-1}(A_{k-1} - l_{k-1} + 1) \end{pmatrix}}_{III} \\
&\times_{i \neq k-1} \underbrace{\begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i + l_i - 1) & \cdots & -\zeta_i(A_i - l_i + 1) \end{pmatrix}}_{I_i} \times \underbrace{\begin{pmatrix} \zeta_k(B_k + l_k) & \cdots & \zeta_k(B_k + l_k - t + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k - l_k) & \cdots & \zeta_k(A_k - l_k - t + 1) \end{pmatrix}}_{II} \\
&\times \underbrace{\begin{pmatrix} \zeta_{k-1}(A_{k-1} - l_{k-1} + 1 + T) & \cdots & \zeta_{k-1}(A_{k-1} - l_{k-1} + 2) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix}}_{(IV)^\vee}
\end{aligned}$$

$$\begin{aligned} & \times \pi_{M, >'_\psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1} - l_{k-1}, B_{k-1} + l_{k-1}, 0, \eta'_{k-1}, \zeta_{k-1}), \right. \\ & \left. (\rho, A_k - l_k - t, B_k + l_k - t, 0, \eta'_k, \zeta_k) \right). \end{aligned}$$

As before, one can show $(IV)^\vee$ are interchangeable with (II) and (I_i) . Then

$$\begin{aligned} \pi_{M, >'_\psi}^{\Sigma_0}(\psi^T, \underline{l}', \underline{\eta}') & \hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T) & \cdots & -\zeta_{k-1}A_{k-1} \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1} + l_{k-1} - 1 + T) & \cdots & -\zeta_{k-1}(A_{k-1} - l_{k-1} + 1) \end{pmatrix} \\ & \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1} + l_{k-1} + T) & \cdots & \zeta_{k-1}(B_{k-1} + l_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} - l_{k-1} + T) & \cdots & \zeta_{k-1}(A_{k-1} - l_{k-1} + 1) \end{pmatrix}}_{III} \\ & \times \underbrace{\begin{pmatrix} \zeta_{k-1}(A_{k-1} - l_{k-1} + 1 + T) & \cdots & \zeta_{k-1}(A_{k-1} - l_{k-1} + 2) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix}}_{(IV)^\vee} \\ & \times_{i \neq k-1} \underbrace{\begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i + l_i - 1) & \cdots & -\zeta_i(A_i - l_i + 1) \end{pmatrix}}_{I_i} \times \underbrace{\begin{pmatrix} \zeta_k(B_k + l_k) & \cdots & \zeta_k(B_k + l_k - t + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k - l_k) & \cdots & \zeta_k(A_k - l_k - t + 1) \end{pmatrix}}_{II} \\ & \times \pi_{M, >'_\psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1} - l_{k-1}, B_{k-1} + l_{k-1}, 0, \eta'_{k-1}, \zeta_{k-1}), \right. \\ & \left. (\rho, A_k - l_k - t, B_k + l_k - t, 0, \eta'_k, \zeta_k) \right). \end{aligned}$$

This implies

$$\begin{aligned} \pi_{M, >'_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') & \hookrightarrow \begin{pmatrix} \zeta_{k-1}B_{k-1} & \cdots & -\zeta_{k-1}A_{k-1} \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1} + l_{k-1} - 1) & \cdots & -\zeta_{k-1}(A_{k-1} - l_{k-1} + 1) \end{pmatrix} \\ & \times_{i \neq k-1} \underbrace{\begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i + l_i - 1) & \cdots & -\zeta_i(A_i - l_i + 1) \end{pmatrix}}_{I_i} \times \underbrace{\begin{pmatrix} \zeta_k(B_k + l_k) & \cdots & \zeta_k(B_k + l_k - t + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k - l_k) & \cdots & \zeta_k(A_k - l_k - t + 1) \end{pmatrix}}_{II} \\ & \times \pi_{M, >'_\psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1} - l_{k-1}, B_{k-1} + l_{k-1}, 0, \eta'_{k-1}, \zeta_{k-1}), \right. \\ & \left. (\rho, A_k - l_k - t, B_k + l_k - t, 0, \eta'_k, \zeta_k) \right). \end{aligned}$$

One can further show $\pi_{M, >'_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}')$ is the unique irreducible subrepresentation. On the other hand,

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) & \hookrightarrow \times_i \underbrace{\begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i + l_i - 1) & \cdots & -\zeta_i(A_i - l_i + 1) \end{pmatrix}}_{I_i} \times \underbrace{\begin{pmatrix} \zeta_k(B_k + l_k) & \cdots & \zeta_k(B_k + l_k - t + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k - l_k) & \cdots & \zeta_k(A_k - l_k - t + 1) \end{pmatrix}}_{II} \\ & \times \pi_{M, >_\psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1} - l_{k-1}, B_{k-1} + l_{k-1}, 0, \eta_{k-1}, \zeta_{k-1}), \right. \end{aligned}$$

$$(\rho, A_k - l_k - t, B_k + l_k - t, 0, \eta_k, \zeta_k)).$$

By (5.5)_r,

$$\begin{aligned} & \pi_{M, >_{\psi}}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1} - l_{k-1}, B_{k-1} + l_{k-1}, 0, \eta'_{k-1}, \zeta_{k-1}), (\rho, A_k - l_k - t, B_k + l_k - t, 0, \eta'_k, \zeta_k) \right) \\ &= \pi_{M, >_{\psi}}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1} - l_{k-1}, B_{k-1} + l_{k-1}, 0, \eta_{k-1}, \zeta_{k-1}), (\rho, A_k - l_k - t, B_k + l_k - t, 0, \eta_k, \zeta_k) \right). \end{aligned}$$

Hence $\pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, >_{\psi}}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}')$. This finishes the second reduction.

5.2.3. Final resolution. Now we want to resolve the case (5.5)_r. Since $l_i = 0$, we can also view ψ as an elementary parameter, denoted by ψ_e . The function $\underline{\eta}$ over $Jord(\psi)$ determines a function ε_e over $Jord(\psi_e)$, i.e., for $C_i \in [A_i, B_i]$,

$$\varepsilon_e(\rho, C_i, C_i, \zeta_i) = \eta_i(-1)^{C_i - B_i}.$$

Similarly, we can define ε'_e . It is obvious that

$$\pi_{M, >_{\psi}}^{\Sigma_0}(\psi, 0, \underline{\eta}) = \pi_{M, >_{\psi_e}}^{\Sigma_0}(\psi_e, \varepsilon_e).$$

Let ψ^T be obtained by changing $(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})$ to $(\rho, A_{k-1} + T, B_{k-1} + T, \zeta_{k-1})$ such that

$$B_{k-1} + T > A_k, \text{ and } B_{k+1} > A_{k-1} + T.$$

Then

$$\pi_{M, >_{\psi}}^{\Sigma_0}(\psi, 0, \underline{\eta}') = \text{Jac}_{(\rho, A_{k-1}+T, B_{k-1}+T, \zeta_{k-1}) \mapsto (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})} \pi_{M, >_{\psi}}^{\Sigma_0}(\psi^T, 0, \underline{\eta}')$$

The order $>_{\psi}$ induces an order $>_{\psi_e}'$ on $Jord(\psi_e)$, and we define

$$\pi_{M, >_{\psi_e}'}^{\Sigma_0}(\psi_e, \varepsilon'_e) := \circ_{C_{k-1} \in [B_{k-1}, A_{k-1}]} \text{Jac}_{(\rho, C_{k-1}+T, C_{k-1}+T, \zeta_{k-1}) \mapsto (\rho, C_{k-1}, C_{k-1}, \zeta_{k-1})} \pi_{M, >_{\psi_e}'}^{\Sigma_0}(\psi_e^T, \varepsilon'_e)$$

Since

$$\pi_{M, >_{\psi_e}'}^{\Sigma_0}(\psi_e^T, \varepsilon'_e) = \pi_{M, >_{\psi}}^{\Sigma_0}(\psi^T, 0, \underline{\eta}'),$$

and

$$\text{Jac}_{(\rho, A_{k-1}+T, B_{k-1}+T, \zeta_{k-1}) \mapsto (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})} = \circ_{C_{k-1} \in [B_{k-1}, A_{k-1}]} \text{Jac}_{(\rho, C_{k-1}+T, C_{k-1}+T, \zeta_{k-1}) \mapsto (\rho, C_{k-1}, C_{k-1}, \zeta_{k-1})}$$

we get

$$\pi_{M, >_{\psi}}^{\Sigma_0}(\psi, 0, \underline{\eta}') = \pi_{M, >_{\psi_e}'}^{\Sigma_0}(\psi_e, \varepsilon'_e).$$

So it is enough to show $\pi_{M, >_{\psi_e}}^{\Sigma_0}(\psi_e, \varepsilon_e) = \pi_{M, >_{\psi_e}'}^{\Sigma_0}(\psi_e, \varepsilon'_e)$. Note

$$\begin{aligned} \pi_{M, >_{\psi_e}}^{\Sigma_0}(\psi_e, \varepsilon_e) &= \pi_W^{\Sigma_0}(\psi_e, \varepsilon_e \varepsilon_{\psi_e}^{M/W}) \\ \pi_{M, >_{\psi_e}'}^{\Sigma_0}(\psi_e, \varepsilon'_e) &= \pi_W^{\Sigma_0}(\psi_e, \varepsilon'_e \varepsilon_{\psi_e}^{M/W}), \end{aligned}$$

where $\varepsilon_{\psi_e}^{M/W}$ (resp. $\varepsilon_{\psi_e}^{M/W}$) is defined with respect to the order $>_{\psi_e}$ (resp. $>_{\psi_e}'$). Then we just need to verify

$$\varepsilon_e \varepsilon_{\psi_e}^{M/W} = \varepsilon'_e \varepsilon_{\psi_e}^{M/W},$$

or equivalently,

$$(5.6) \quad \frac{\varepsilon_e}{\varepsilon'_e} = \frac{\varepsilon_{\psi_e}^{M/W}}{\varepsilon_{\psi_e}^{M/W}} = \frac{\varepsilon_{\psi_e}^{M/MW}}{\varepsilon_{\psi_e}^{M/MW}} \cdot \frac{\varepsilon_{\psi_e}^{MW/W}}{\varepsilon_{\psi_e}^{MW/W}}.$$

We divide it into two cases:

(1) If $A_i \in \mathbb{Z}$,

$$\varepsilon_{\psi_e}^{MW/W}(\rho, C_i, C_i, \zeta_i) = \varepsilon_{\psi_e}'^{MW/W}(\rho, C_i, C_i, \zeta_i) = 1.$$

And

$$\varepsilon_{\psi_e}^{M/MW}(\rho, C_i, C_i, \zeta_i) = \begin{cases} (-1)^m & \text{if } \zeta_i = +1, \\ (-1)^{m+n} & \text{if } \zeta_i = -1, \end{cases}$$

where

$$m = \#\{C_j \in [A_j, B_j] \text{ for all } j : \zeta_j = -1, (\rho, C_j, C_j, \zeta_j) >_{\psi_e} (\rho, C_i, C_i, \zeta_i)\},$$

$$n = \#\{C_j \in [A_j, B_j] \text{ for all } j : (\rho, C_i, C_i, \zeta_i) >_{\psi_e} (\rho, C_j, C_j, \zeta_j)\}.$$

And

$$\varepsilon_{\psi_e}'^{M/MW}(\rho, C_i, C_i, \zeta_i) = \begin{cases} (-1)^{m'} & \text{if } \zeta_i = +1, \\ (-1)^{m'+n'} & \text{if } \zeta_i = -1, \end{cases}$$

where

$$m' = \#\{C_j \in [A_j, B_j] \text{ for all } j : \zeta_j = -1, (\rho, C_j, C_j, \zeta_j) >_{\psi_e}' (\rho, C_i, C_i, \zeta_i)\},$$

$$n' = \#\{C_j \in [A_j, B_j] \text{ for all } j : (\rho, C_i, C_i, \zeta_i) >_{\psi_e}' (\rho, C_j, C_j, \zeta_j)\}.$$

(a) $i \neq k, k-1$

- $\varepsilon_e(\rho, C_i, C_i, \zeta_i)/\varepsilon_e'(\rho, C_i, C_i, \zeta_i) = 1$
- $\varepsilon_{\psi_e}^{M/MW}(\rho, C_i, C_i, \zeta_i)/\varepsilon_{\psi_e}'^{M/MW}(\rho, C_i, C_i, \zeta_i) = 1$

(b) $i = k$

- $\varepsilon_e(\rho, C_k, C_k, \zeta_k)/\varepsilon_e'(\rho, C_k, C_k, \zeta_k) = (-1)^{A_{k-1}-B_{k-1}+1}$
- $\varepsilon_{\psi_e}^{M/MW}(\rho, C_k, C_k, \zeta_k)/\varepsilon_{\psi_e}'^{M/MW}(\rho, C_k, C_k, \zeta_k) = (-1)^{A_{k-1}-B_{k-1}+1}$

(c) $i = k-1$

- $\varepsilon_e(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1})/\varepsilon_e'(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1}) = (-1)^{A_k-B_k+1}$
- $\varepsilon_{\psi_e}^{M/MW}(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1})/\varepsilon_{\psi_e}'^{M/MW}(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1}) = (-1)^{A_k-B_k+1}$

(2) $A_i \notin \mathbb{Z}$

$$\varepsilon_{\psi_e}^{M/MW}(\rho, C_i, C_i, \zeta_i) = \varepsilon_{\psi_e}'^{M/MW}(\rho, C_i, C_i, \zeta_i) = 1.$$

And

$$\varepsilon_{\psi_e}^{MW/W}(\rho, C_i, C_i, \zeta_i) = \begin{cases} (-1)^m & \text{if } \zeta_i = +1, \\ (-1)^n & \text{if } \zeta_i = -1, \end{cases}$$

where

$$m = \#\{C_j \in [A_j, B_j] \text{ for all } j : \zeta_j = -1, (\rho, C_j, C_j, \zeta_j) >_{\psi_e} (\rho, C_i, C_i, \zeta_i)\},$$

$$n = \#\{C_j \in [A_j, B_j] \text{ for all } j : \zeta_j = +1, (\rho, C_i, C_i, \zeta_i) >_{\psi_e} (\rho, C_j, C_j, \zeta_j)\}.$$

And

$$\varepsilon_{\psi_e}'^{MW/W}(\rho, C_i, C_i, \zeta_i) = \begin{cases} (-1)^{m'} & \text{if } \zeta_i = +1, \\ (-1)^{n'} & \text{if } \zeta_i = -1, \end{cases}$$

where

$$m' = \sharp\{C_j \in [A_j, B_j] \text{ for all } j : \zeta_j = -1, (\rho, C_j, C_j, \zeta_j) >_{\psi_e}' (\rho, C_i, C_i, \zeta_i)\},$$

$$n' = \sharp\{C_j \in [A_j, B_j] \text{ for all } j : \zeta_j = +1, (\rho, C_i, C_i, \zeta_i) >_{\psi_e}' (\rho, C_j, C_j, \zeta_j)\}.$$

(a) $i \neq k, k-1$

- $\varepsilon_e(\rho, C_i, C_i, \zeta_i) / \varepsilon_e'(\rho, C_i, C_i, \zeta_i) = 1$
- $\varepsilon_{\psi_e}^{MW/W}(\rho, C_i, C_i, \zeta_i) / \varepsilon_{\psi_e}'^{MW/W}(\rho, C_i, C_i, \zeta_i) = 1$

(b) $i = k$

- $\varepsilon_e(\rho, C_k, C_k, \zeta_k) / \varepsilon_e'(\rho, C_k, C_k, \zeta_k) = (-1)^{A_{k-1}-B_{k-1}+1}$
- $\varepsilon_{\psi_e}^{MW/W}(\rho, C_k, C_k, \zeta_k) / \varepsilon_{\psi_e}'^{MW/W}(\rho, C_k, C_k, \zeta_k) = (-1)^{A_{k-1}-B_{k-1}+1}$

(c) $i = k-1$

- $\varepsilon_e(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1}) / \varepsilon_e'(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1}) = (-1)^{A_k-B_k+1}$
- $\varepsilon_{\psi_e}^{MW/W}(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1}) / \varepsilon_{\psi_e}'^{MW/W}(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1}) = (-1)^{A_k-B_k+1}$

It follows from the calculations above that (5.6) holds, and this ends the proof of Proposition 5.3.

6. REDUCTION OPERATIONS

In this section, we want to give three operations, which will be used in our general procedure to reduce the problem of finding nonvanishing conditions for $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$.

6.1. Pull.

6.1.1. *Case of unequal length.* We choose an admissible order $>_\psi$, and we also fix a self-dual unitary irreducible supercuspidal representation ρ of $GL(d_\rho)$. We index the Jordan blocks in $Jord_\rho(\psi)$ such that

$$(\rho, A_{i+1}, B_{i+1}, \zeta_{i+1}) >_\psi (\rho, A_i, B_i, \zeta_i).$$

Suppose there exists n such that for $i > n$,

$$(\rho, A_i, B_i, \zeta_i) \gg \cup_{j=1}^n \{(\rho, A_j, B_j, \zeta_j)\}.$$

Moreover

$$[A_n, B_n] \supsetneq [A_{n-1}, B_{n-1}] \text{ and } \zeta_n = \zeta_{n-1}.$$

We denote by $>_\psi'$ the order obtained from $>_\psi$ by switching $(\rho, A_n, B_n, \zeta_n)$ and $(\rho, A_{n-1}, B_{n-1}, \zeta_{n-1})$. Let S_n^+ be the corresponding transformation on $(\underline{l}, \underline{\eta})$. We define ψ_- by

$$Jord(\psi_-) = Jord(\psi) \setminus \{(\rho, A_n, B_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1})\}.$$

We denote the restriction of $(\underline{l}, \underline{\eta})$ to $Jord(\psi_-)$ by $(\underline{l}_-, \underline{\eta}_-)$.

Proposition 6.1. *For any $(\underline{l}, \underline{\eta})$, $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if the following conditions are satisfied:*

(1)

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n, B_n + T_n, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + T_{n-1}, B_{n-1} + T_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0$$

for any T_n, T_{n-1} , such that

$$[A_n + T_n, B_n + T_n] \supsetneq [A_{n-1} + T_{n-1}, B_{n-1} + T_{n-1}]$$

and $(\rho, A_i, B_i, \zeta_i) \gg (\rho, A_n + T, B_n + T, \zeta_n)$ for $i > n$.

(2)

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0$$

for any T such that $B_i > A_n + T$ for $i > n$.

(3)

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n, B_n, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1} + T, B_{n-1} + T, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \neq 0$$

for any T such that $B_i > A_{n-1} + T$ for $i > n$, and $(\underline{l}', \underline{\eta}') = S_n^+(\underline{l}, \underline{\eta})$.

Proof. Let $\zeta = \zeta_n = \zeta_{n-1}$, and $(\underline{l}', \underline{\eta}') = S_n^+(\underline{l}, \underline{\eta})$ as in the proposition. Since $[A_n, B_n] \supsetneq [A_{n-1}, B_{n-1}]$, we necessarily have $[A_n + 1, B_n + 1] \supsetneq [A_{n-1}, B_{n-1}]$ or $[A_n, B_n] \supsetneq [A_{n-1} + 1, B_{n-1} + 1]$. So we divide it into two cases.

Suppose $[A_n + 1, B_n + 1] \supsetneq [A_{n-1}, B_{n-1}]$, we claim $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if the following conditions are satisfied:

•

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n + 1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0;$$

•

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n, B_n, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1} + T, B_{n-1} + T, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \neq 0$$

for any T such that $B_i > A_{n-1} + T$ for $i > n$.

The necessity of these conditions is obvious, so we will only need to show its sufficiency. We can take T sufficiently large such that $B_i > A_{n-1} + T$ for $i > n$, and

$$(\rho, A_{n-1} + T, B_{n-1} + T, \zeta_{n-1}) \gg \cup_{j=1}^{n-2} \{(\rho, A_j, B_j, \zeta_j)\} \cup \{(\rho, A_n + 1, B_n + 1, \zeta_n)\}.$$

By Proposition 5.1, we have

$$\begin{aligned} & \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n + 1, B_n + 1, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) = \\ & \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n + 1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0. \end{aligned}$$

So

$$\pi_{\gg}^{\Sigma_0} := \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n + 1, B_n + 1, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1} + T, B_{n-1} + T, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \neq 0$$

and

$$\begin{aligned} \pi_{\gg}^{\Sigma_0} & \hookrightarrow \begin{pmatrix} \zeta(B_{n-1} + T) & \cdots & \zeta(B_{n-1} + 1) \\ \vdots & & \vdots \\ \zeta(A_{n-1} + T) & \cdots & \zeta(A_{n-1} + 1) \end{pmatrix} \\ & \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n + 1, B_n + 1, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \\ & = \begin{pmatrix} \zeta(B_{n-1} + T) & \cdots & \zeta(B_{n-1} + 1) \\ \vdots & & \vdots \\ \zeta(A_{n-1} + T) & \cdots & \zeta(A_{n-1} + 1) \end{pmatrix} \\ & \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n + 1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})). \end{aligned}$$

Note

$$\begin{aligned} & \text{Jac}_{(\rho, A_n+1, B_n+1, \zeta) \mapsto (\rho, A_n, B_n, \zeta)} \pi_{\gg}^{\Sigma_0} = \\ & \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n, B_n, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1} + T, B_{n-1} + T, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \neq 0. \end{aligned}$$

So after we apply the same Jacquet functor to the full induced representation above, we should get something nonzero. To compute this Jacquet module, one notes

$$\zeta(B_{n-1} + T), -\zeta(A_{n-1} + 1) \notin \{\zeta(B_n + 1), \dots, \zeta(A_n + 1)\},$$

so it can only be

$$\begin{aligned} & \begin{pmatrix} \zeta(B_{n-1} + T) & \cdots & \zeta(B_{n-1} + 1) \\ \vdots & & \vdots \\ \zeta(A_{n-1} + T) & \cdots & \zeta(A_{n-1} + 1) \end{pmatrix} \times \text{Jac}_{(\rho, A_n+1, B_n+1, \zeta) \mapsto (\rho, A_n, B_n, \zeta)} \\ & \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n + 1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \\ & = \begin{pmatrix} \zeta(B_{n-1} + T) & \cdots & \zeta(B_{n-1} + 1) \\ \vdots & & \vdots \\ \zeta(A_{n-1} + T) & \cdots & \zeta(A_{n-1} + 1) \end{pmatrix} \times \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0. \end{aligned}$$

Hence $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$. This shows our claim in the first case.

Suppose $[A_n, B_n] \supsetneq [A_{n-1} + 1, B_{n-1} + 1]$, we claim $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if the following conditions are satisfied:

- $\pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n, B_n, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \neq 0;$
- $\pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0$
for any T such that $B_i > A_n + T$ for $i > n$.

The argument of this case is essentially the same as the previous one. The necessity of these condition is again obvious, so it suffices to show their sufficiency. We can take T sufficiently large such that $B_i > A_n + T$ for $i > n$, and

$$(\rho, A_n + T, B_n + T, \zeta_n) \gg \cup_{j=1}^{n-2} \{(\rho, A_j, B_j, \zeta_j)\} \cup \{(\rho, A_{n-1} + 1, B_{n-1} + 1, \zeta_{n-1})\}.$$

By Proposition 5.1, we have

$$\begin{aligned} & \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n, B_n, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) = \\ & \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n, B_n, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \neq 0. \end{aligned}$$

So

$$\pi_{\gg}^{\Sigma_0} := \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0$$

and

$$\begin{aligned} \pi_{\gg}^{\Sigma_0} & \hookrightarrow \begin{pmatrix} \zeta(B_n + T) & \cdots & \zeta(B_n + 1) \\ \vdots & & \vdots \\ \zeta(A_n + T) & \cdots & \zeta(A_n + 1) \end{pmatrix} \\ & \times \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n, B_n, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \\ & = \begin{pmatrix} \zeta(B_n + T) & \cdots & \zeta(B_n + 1) \\ \vdots & & \vdots \\ \zeta(A_n + T) & \cdots & \zeta(A_n + 1) \end{pmatrix} \\ & \times \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n, B_n, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})). \end{aligned}$$

Note

$$\text{Jac}_{(\rho, A_{n-1}+1, B_{n-1}+1, \zeta) \mapsto (\rho, A_{n-1}, B_{n-1}, \zeta)} \pi_{\gg}^{\Sigma_0} =$$

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0.$$

So after we apply the same Jacquet functor to the full induced representation above, we should get something nonzero. To compute this Jacquet module, one notes

$$\zeta(B_n + T), -\zeta(A_n + 1) \notin \{\zeta(B_{n-1} + 1), \dots, \zeta(A_{n-1} + 1)\},$$

so it can only be

$$\begin{aligned} & \begin{pmatrix} \zeta(B_n + T) & \cdots & \zeta(B_n + 1) \\ \vdots & & \vdots \\ \zeta(A_n + T) & \cdots & \zeta(A_n + 1) \end{pmatrix} \times \text{Jac}_{(\rho, A_{n-1}+1, B_{n-1}+1, \zeta) \mapsto (\rho, A_{n-1}, B_{n-1}, \zeta)} \\ & \pi_{M, >'_\psi}^{\Sigma_0}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n, B_n, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \\ & = \begin{pmatrix} \zeta(B_n + T) & \cdots & \zeta(B_n + 1) \\ \vdots & & \vdots \\ \zeta(A_n + T) & \cdots & \zeta(A_n + 1) \end{pmatrix} \times \pi_{M, >'_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') \neq 0. \end{aligned}$$

Hence $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, >'_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') \neq 0$. This shows our claim in the second case.

By combining our claims in both cases in some alternative way, we can shift both $[A_n, B_n], [A_{n-1}, B_{n-1}]$ to $[A_n + T_n, B_n + T_n], [A_{n-1} + T_{n-1}, B_{n-1} + T_{n-1}]$ for any T_n, T_{n-1} such that

$$[A_n + T_n, B_n + T_n] \supsetneq [A_{n-1} + T_{n-1}, B_{n-1} + T_{n-1}]$$

and $(\rho, A_i, B_i, \zeta_i) \gg (\rho, A_n + T_n, B_n + T_n, \zeta_n)$ for $i > n$. Then the statement of this proposition is clear. \square

Remark 6.2. The way we will use this proposition is to take all T_n, T_{n-1} and T to be large.

6.1.2. Case of equal length. We choose an admissible order $>_\psi$, and we also fix a self-dual unitary irreducible supercuspidal representation ρ of $GL(d_\rho)$. We index the Jordan blocks in $Jord_\rho(\psi)$ such that

$$(\rho, A_{i+1}, B_{i+1}, \zeta_{i+1}) >_\psi (\rho, A_i, B_i, \zeta_i).$$

Suppose there exists n such that for $i > n$,

$$(\rho, A_i, B_i, \zeta_i) \gg \cup_{j=1}^n \{(\rho, A_j, B_j, \zeta_j)\}.$$

Moreover

$$[A_n, B_n] = [A_{n-1}, B_{n-1}] \text{ and } \zeta_n = \zeta_{n-1}.$$

And there exists no $[A_i, B_i] \subsetneq [A_n, B_n]$ with $\zeta_i = \zeta_n$ for $i < n$. We define ψ_- by

$$Jord(\psi_-) = Jord(\psi) \setminus \{(\rho, A_n, B_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1})\}.$$

We denote the restriction of $(\underline{l}, \underline{\eta})$ to $Jord(\psi_-)$ by $(\underline{l}_-, \underline{\eta}_-)$.

Proposition 6.3. *For any $(\underline{l}, \underline{\eta})$, $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if the following conditions are satisfied:*

(1)

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n, B_n + T_n, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + T_{n-1}, B_{n-1} + T_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0$$

for $T_n = T_{n-1}$ such that $(\rho, A_i, B_i, \zeta_i) \gg (\rho, A_n + T_n, B_n + T_n, \zeta_n)$ for $i > n$.

(2)

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0$$

for any T such that $B_i > A_n + T$ for $i > n$.

Proof. For simplicity, let $[A_n, B_n] = [A, B]$ and $\zeta_n = \zeta$. The necessity of these conditions are clear, so it suffices to show their sufficiency. In fact it is enough to prove the proposition by letting $T_n = T_{n-1} = 1$ in the first condition. So let us suppose

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n + 1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0.$$

We can take T sufficiently large such that $B_i > A_n + T$ for $i > n$, and

$$(\rho, A_n + T, B_n + T, \zeta_n) \gg \cup_{j=1}^{n-2} \{(\rho, A_j, B_j, \zeta_j)\} \cup \{(\rho, A_{n-1} + 1, B_{n-1} + 1, \zeta_{n-1})\}.$$

Let

$$\pi_{\gg}^{\Sigma_0} := \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})).$$

Then

$$\begin{aligned} \pi_{\gg}^{\Sigma_0} &\hookrightarrow \begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix} \\ &\rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n + 1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \end{aligned}$$

Since

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0,$$

then

$$(6.1) \quad \text{Jac}_{\zeta(B+1), \dots, \zeta(A+1)} \pi_{\gg}^{\Sigma_0} \neq 0.$$

Note $\zeta(B+T) \notin [\zeta(B+1), \zeta(A+1)]$. So this implies

$$\begin{aligned} &\text{Jac}_{\zeta(B+1), \dots, \zeta(A+1)} \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n + 1, l_n, \eta_n, \zeta_n), \\ &(\rho, A_{n-1} + 1, B_{n-1} + 1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0 \end{aligned}$$

Then there exists $C \in [A+1, B+1]$ and an irreducible representation σ such that

$$\begin{aligned} &\pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n + 1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \\ &\hookrightarrow \langle \zeta C, \dots, \zeta(A+1) \rangle \rtimes \sigma. \end{aligned}$$

Since there exists no $[A_i, B_i] \subsetneq [A_n, B_n]$ with $\zeta_i = \zeta_n$ for $i < n$, we must have $C = B+1$. Therefore,

$$\pi_{\gg}^{\Sigma_0} \hookrightarrow \begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix} \times \begin{pmatrix} \zeta(B+1) \\ \vdots \\ \zeta(A+1) \end{pmatrix} \rtimes \sigma$$

Let us denote the full induced representation above by $(*-1)$. By Frobenius reciprocity, σ is an irreducible constituent of

$$\text{Jac}_{\zeta(B+1), \dots, \zeta(A+1)} \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n + 1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})).$$

In fact it is not hard to show the Jacquet module above consists of representations in

$$\Pi^{\Sigma_0}(\psi_-, (\rho, A_n + 1, B_n + 1, \zeta_n), (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1}))$$

with multiplicities. So in particular, σ is an element in the above packet. We claim

$$(6.2) \quad \text{Jac}_{\zeta(B+1), \dots, \zeta(A+1)} \sigma \neq 0.$$

Otherwise, one finds

$$\text{Jac}_{\zeta C', \dots, \zeta C''} \sigma = 0$$

for any $C' \in [B+1, A+T]$ and $C'' \in [A+1, A+T]$. This implies

$$\text{Jac}_{(\rho, A+T, B+T, \zeta) \mapsto (\rho, A, B, \zeta)} (*-1) = \sigma.$$

So $(* - 1)$ has a unique irreducible subrepresentation, and hence

$$\pi_{\gg}^{\Sigma_0} \hookrightarrow \begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+1) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+1) \end{pmatrix} \rtimes \sigma$$

Then $\text{Jac}_{\zeta(B+1), \dots, \zeta(A+1)} \sigma = 0$ implies $\text{Jac}_{\zeta(B+1), \dots, \zeta(A+1)} \pi_{\gg}^{\Sigma_0} = 0$, but this contradicts to (6.1). Now by (6.2), we have

$$\sigma \hookrightarrow \begin{pmatrix} \zeta C \\ \vdots \\ \zeta(A+1) \end{pmatrix} \rtimes \sigma'$$

for some $C \in [B+1, A+1]$ and some irreducible representation σ' . For the same reason as before, we must have $C = B+1$. This also implies $\sigma' \in \Pi_{\psi}^{\Sigma_0}$. Therefore

$$\pi_{\gg}^{\Sigma_0} \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix} \times \begin{pmatrix} \zeta(B+1) \\ \vdots \\ \zeta(A+1) \end{pmatrix} \times \begin{pmatrix} \zeta(B+1) \\ \vdots \\ \zeta(A+1) \end{pmatrix}}_{(*-2)} \rtimes \sigma'$$

There exists an irreducible constituent τ of $(*-2)$ such that

$$\pi_{\gg}^{\Sigma_0} \hookrightarrow \tau \rtimes \sigma'.$$

Since $\text{Jac}_x \sigma' = 0$ for any $x \in [\zeta(A+1), \zeta(B+1)]$, we can conclude from (6.1) that

$$\text{Jac}_{\zeta(B+1), \dots, \zeta(A+1)} \tau \neq 0.$$

So there exists $C \in [B+1, A+1]$ and an irreducible representation τ' such that

$$\tau \hookrightarrow \begin{pmatrix} \zeta C \\ \vdots \\ \zeta(A+1) \end{pmatrix} \rtimes \tau'$$

From $(*-2)$, we see C can only be $B+1$. Hence, τ' is an irreducible constituent of

$$\text{Jac}_{\zeta(B+1), \dots, \zeta(A+1)} (*-2) = 2 \cdot \begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix} \times \begin{pmatrix} \zeta(B+1) \\ \vdots \\ \zeta(A+1) \end{pmatrix}$$

If $\text{Jac}_{\zeta(B+1)} \tau' \neq 0$, then $\text{Jac}_{\zeta(B+1), \zeta(B+1)} \pi_{\gg}^{\Sigma_0} \neq 0$. This is only possible when there exists $(\rho, A_i, B_i, \zeta_i) \in \text{Jord}(\psi)$ such that $B_i = B+1$ and $\zeta_i = \zeta$. But then necessarily $A_i \leq A_n$ and $[A_i, B_i] \subsetneq [A_n, B_n]$, which is excluded by our assumption. Therefore, we must have

$$\tau' = \begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+1) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+1) \end{pmatrix}.$$

To summarize, we get

$$\pi_{\gg}^{\Sigma_0} \hookrightarrow \begin{pmatrix} \zeta(B+1) \\ \vdots \\ \zeta(A+1) \end{pmatrix} \times \begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+1) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+1) \end{pmatrix} \rtimes \sigma'.$$

Hence,

$$\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \text{Jac}_{(\rho, A+T, B+T, \zeta) \mapsto (\rho, A, B, \zeta)} \circ \text{Jac}_{\zeta(B+1), \dots, \zeta(A+1)} \pi_{\gg}^{\Sigma_0} \neq 0.$$

This finishes the proof.

□

6.2. Expand. We choose an admissible order $>_\psi$, and we also fix a self-dual unitary irreducible supercuspidal representation ρ of $GL(d_\rho)$. We index the Jordan blocks in $Jord_\rho(\psi)$ such that

$$(\rho, A_{i+1}, B_{i+1}, \zeta_{i+1}) >_\psi (\rho, A_i, B_i, \zeta_i).$$

Suppose there exists n such that for $i > n$,

$$(\rho, A_i, B_i, \zeta_i) \gg_2 \cup_{j=1}^n \{(\rho, A_j, B_j, \zeta_j)\}.$$

Moreover, for $i < n$,

$$A_n \geq A_i \text{ and there exists no } [A_i, B_i] \subseteq [A_n, B_n] \text{ with } \zeta_i = \zeta_n.$$

Let t_n be the smallest integer such that $B_n - t_n = B_i$ for some $i < n$ and $\zeta_i = \zeta_n$. If such t_n does not exist, we let $t_n := [B_n]$. We define ψ_- by

$$Jord(\psi_-) = Jord(\psi) \setminus \{(\rho, A_n, B_n, \zeta_n)\}.$$

We denote the restriction of $(\underline{l}, \underline{\eta})$ to $Jord(\psi_-)$ by $(\underline{l}_-, \underline{\eta}_-)$.

Proposition 6.4. *For any $(\underline{l}, \underline{\eta})$ and positive integer $t \leq t_n$, $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if*

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + t, B_n - t, l_n + t, \eta_n, \zeta_n)) \neq 0.$$

Moreover,

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + t, B_n - t, l_n + t, \eta_n, \zeta_n)) \hookrightarrow \\ \left(\begin{array}{ccc} \zeta_n(B_n - t) & \cdots & -\zeta_n(A_n + t) \\ \vdots & & \vdots \\ \zeta_n(B_n - 1) & \cdots & -\zeta_n(A_n + 1) \end{array} \right) \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}). \end{aligned}$$

as the unique irreducible subrepresentation, and

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \circ_{i \in [1, t]} \text{Jac}_{\zeta_n(B_n - i), \dots, -\zeta_n(A_n + i)} \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + t, B_n - t, l_n + t, \eta_n, \zeta_n)).$$

Proof. We will first consider the case $t = 1$. Let ψ_{\gg} dominates ψ with discrete diagonal restriction such that

$$(\rho, A_{n+1}, B_{n+1}, \zeta_{n+1}) \gg (\rho, A_n + T_n + 1, B_n + T_n - 1, \zeta_n) \gg (\rho, A_{n-1} + T_{n-1}, B_n + T_{n-1}, \zeta_{n-1}).$$

Let $\psi_{\gg, -}$ be obtained from ψ_{\gg} by removing $(\rho, A_n + T_n, B_n + T_n, \zeta_n)$. Then

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n + 1, B_n + T_n - 1, l_n + 1, \eta_n, \zeta_n)) \hookrightarrow \\ < \zeta_n(B_n + T_n - 1), \dots, -\zeta_n(A_n + T_n + 1) > \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n, B_n + T_n, l_n, \eta_n, \zeta_n)). \end{aligned}$$

Suppose

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n - 1, l_n + 1, \eta_n, \zeta_n)) \neq 0.$$

Let

$$\begin{aligned} \text{Jac}_{X_{>n}} &:= \circ_{i>n} \text{Jac}_{(\rho, A_i+T_i, B_i+T_i, \zeta_i) \mapsto (\rho, A_i, B_i, \zeta_i)} \\ \text{Jac}_{X'_n} &:= \text{Jac}_{(\rho, A_n+T_n+1, B_n+T_n-1, \zeta_n) \mapsto (\rho, A_n+1, B_n-1, \zeta_n)} \\ \text{Jac}_{X_{<n}} &:= \circ_{i<n} \text{Jac}_{(\rho, A_i+T_i, B_i+T_i, \zeta_i) \mapsto (\rho, A_i, B_i, \zeta_i)} \end{aligned}$$

where i decreases. Then after we apply

$$\text{Jac}_{X_{>n}} \circ \text{Jac}_{X'_n} \circ \text{Jac}_{X_{<n}}$$

and Jac_{X^c} to the full induced representation

(6.3)

$$< \zeta_n(B_n + T_n - 1), \dots, -\zeta_n(A_n + T_n + 1) > \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n, B_n + T_n, l_n, \eta_n, \zeta_n)),$$

we should get something nonzero.

For $i < n$, one notes

$$\zeta_n(B_n + T_n - 1), \zeta_n(A_n + T_n + 1) \notin [\zeta_i(A_i + T_i), \zeta_i(B_i + 1)].$$

So $\text{Jac}_{X_{<n}}(6.3)$ becomes

$$\begin{aligned} & < \zeta_n(B_n + T_n - 1), \dots, -\zeta_n(A_n + T_n + 1) > \rtimes \text{Jac}_{X_{<n}} \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n)) \\ & = < \zeta_n(B_n + T_n - 1), \dots, -\zeta_n(A_n + T_n + 1) > \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}^{(n-1)}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n)). \end{aligned}$$

where $\psi_{\gg,-}^{(n-1)}$ is obtained from $\psi_{\gg,-}$ by changing $(\rho, A_i + T_i, B_i + T_i, \zeta_i)$ to $(\rho, A_i, B_i, \zeta_i)$ for $i < n$.

For $i = n$, we can further write $\text{Jac}_{X'_n}$ as

$$\text{Jac}_{\zeta_n(A_n+T_n+1), \dots, \zeta_n(A_n+2)} \circ \text{Jac}_{(\rho, A_n+T_n, B_n+T_n, \zeta_n) \mapsto (\rho, A_n, B_n, \zeta_n)} \circ \text{Jac}_{\zeta_n(B_n+T_n-1), \dots, \zeta_n B_n}$$

First, we claim $\text{Jac}_{\zeta_n(B_n+T_n-1), \dots, \zeta_n B_n}$ can only apply to

$$< \zeta_n(B_n + T_n - 1), \dots, -\zeta_n(A_n + T_n + 1) > .$$

Otherwise, there exists $x \in [\zeta_n(B_n + T_n - 1), \zeta_n B_n]$ such that

$$\text{Jac}_x \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}^{(n-1)}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n)) \neq 0.$$

This can only happen when there exists $i < n$ such that $B_i \geq B_n$ and $\zeta_i = \zeta_n$, but that contradicts to our assumption. As a result,

$$\begin{aligned} & \text{Jac}_{\zeta_n(B_n+T_n-1), \dots, \zeta_n B_n} \circ \text{Jac}_{X_{<n}}(6.3) = < \zeta_n(B_n - 1), \dots, -\zeta_n(A_n + T_n + 1) > \\ & \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}^{(n-1)}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n)). \end{aligned}$$

Secondly, we claim $\text{Jac}_{(\rho, A_n+T_n, B_n+T_n, \zeta_n) \mapsto (\rho, A_n, B_n, \zeta_n)}$ can only apply to

$$\pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}^{(n-1)}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n, B_n + T_n, l_n, \eta_n, \zeta_n)).$$

This is because

$$\zeta_n(B_n - 1), \zeta_n(A_n + T_n + 1) \notin [\zeta_n(A_n + T_n), \zeta_n(B_n + 1)].$$

So

$$\begin{aligned} & \text{Jac}_{(\rho, A_n+T_n, B_n+T_n, \zeta_n) \mapsto (\rho, A_n, B_n, \zeta_n)} \circ \text{Jac}_{\zeta_n(B_n+T_n-1), \dots, \zeta_n(B_n)} \circ \text{Jac}_{X_{<n}}(6.3) = \\ & < \zeta_n(B_n - 1), \dots, -\zeta_n(A_n + T_n + 1) > \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}^{(n-1)}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n, B_n, l_n, \eta_n, \zeta_n)). \end{aligned}$$

Thirdly, $\text{Jac}_{\zeta_n(A_n+T_n+1), \dots, \zeta_n(A_n+2)}$ can only apply to

$$< \zeta_n(B_n - 1), \dots, -\zeta_n(A_n + T_n + 1) >$$

for the same reason as before, so

$$\text{Jac}_{X'_n} \circ \text{Jac}_{X_{<n}}(6.3) = < \zeta_n(B_n - 1), \dots, -\zeta_n(A_n + 1) > \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}^{(n-1)}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n, B_n, l_n, \eta_n, \zeta_n)).$$

For $i > n$, $\text{Jac}_{X_{>n}}$ can only apply to $\pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}^{n-1}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n, B_n, l_n, \eta_n, \zeta_n))$ as $B_i > A_n + 1$. Therefore,

$$\text{Jac}_{X^c} \circ \text{Jac}_{X_{>n}} \circ \text{Jac}_{X'_n} \circ \text{Jac}_{X_{<n}}(6.3) = < \zeta_n(B_n - 1), \dots, -\zeta_n(A_n + 1) > \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0.$$

Hence $\pi_{M,>\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$.

Next, we suppose $\pi_{M,>\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$. Let

$$\mathcal{C}_{X_{>n}} := \times_{i>n} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix},$$

$$\mathcal{C}_{X_{<n}} := \times_{i < n} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix},$$

where i increases, and

$$\mathcal{C}_{X_n} := \begin{pmatrix} \zeta_n(B_n + T_n) & \cdots & \zeta_n(B_n + 1) \\ \vdots & & \vdots \\ \zeta_n(A_n + T_n) & \cdots & \zeta_n(A_n + 1) \end{pmatrix}.$$

Then

$$\begin{aligned} & \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n + 1, B_n + T_n - 1, l_n + 1, \eta_n, \zeta_n)) \hookrightarrow \\ & < \zeta_n(B_n + T_n - 1), \dots, -\zeta_n(A_n + T_n + 1) > \times \mathcal{C}_{X_{<n}} \times \mathcal{C}_{X_n} \times \mathcal{C}_{X_{>n}} \times \mathcal{C}_{X^c} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \cong \\ & \mathcal{C}_{X^c} \times \mathcal{C}_{X_{<n}} \times < \zeta_n(B_n + T_n - 1), \dots, \zeta_n B_n > \times \mathcal{C}_{X_n} \times \mathcal{C}_{X_{>n}} \times < \zeta_n(B_n - 1), \dots, -\zeta_n(A_n + 1) > \times \\ & < -\zeta_n(A_n + 2), \dots, -\zeta_n(A_n + T_n + 1) > \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}). \end{aligned}$$

By Corollary 3.7, we can take the dual of $< -\zeta_n(A_n + 2), \dots, -\zeta_n(A_n + T_n + 1) >$. Hence

$$\begin{aligned} & \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n + 1, B_n + T_n - 1, l_n + 1, \eta_n, \zeta_n)) \hookrightarrow \\ & \mathcal{C}_{X^c} \times \mathcal{C}_{X_{<n}} \times < \zeta_n(B_n + T_n - 1), \dots, \zeta_n B_n > \times \mathcal{C}_{X_n} \times \mathcal{C}_{X_{>n}} \times < \zeta_n(B_n - 1), \dots, -\zeta_n(A_n + 1) > \times \\ & < \zeta_n(A_n + T_n + 1), \dots, \zeta_n(A_n + 2) > \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \cong \\ & \mathcal{C}_{X^c} \times \mathcal{C}_{X_{<n}} \times < \zeta_n(B_n + T_n - 1), \dots, \zeta_n B_n > \times \mathcal{C}_{X_n} \times < \zeta_n(A_n + T_n + 1), \dots, \zeta_n(A_n + 2) > \times \\ & \mathcal{C}_{X_{>n}} \times < \zeta_n(B_n - 1), \dots, -\zeta_n(A_n + 1) > \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \end{aligned}$$

Therefore,

$$\begin{aligned} & \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n - 1, l_n + 1, \eta_n, \zeta_n)) \hookrightarrow \\ & < \zeta_n(B_n - 1), \dots, -\zeta_n(A_n + 1) > \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}). \end{aligned}$$

This proves the proposition in case $t = 1$ except for the uniqueness and the statements about Jacquet modules.

In fact, the first part of the proposition follows easily from that of case $t = 1$. Moreover, we have

$$\begin{aligned} & \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + t, B_n - t, l_n + t, \eta_n, \zeta_n)) \hookrightarrow \\ & < \zeta_n(B_n - t), \dots, -\zeta_n(A_n + t) > \times \cdots \times < \zeta_n(B_n - 1), \dots, -\zeta_n(A_n + 1) > \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}). \end{aligned}$$

Then there exists an irreducible constituent τ of

$$< \zeta_n(B_n - t), \dots, -\zeta_n(A_n + t) > \times \cdots \times < \zeta_n(B_n - 1), \dots, -\zeta_n(A_n + 1) >$$

such that

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + t, B_n - t, l_n + t, \eta_n, \zeta_n)) \hookrightarrow \tau \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}).$$

We claim

$$\tau = \begin{pmatrix} \zeta_n(B_n - t) & \cdots & -\zeta_n(A_n + t) \\ \vdots & & \vdots \\ \zeta_n(B_n - 1) & \cdots & -\zeta_n(A_n + 1) \end{pmatrix}.$$

Otherwise, $\text{Jac}_x \tau \neq 0$ for some x in $]\zeta_n(B_n - t), \zeta_n(B_n - 1)]$, and hence

$$\text{Jac}_x \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + t, B_n - t, l_n + t, \eta_n, \zeta_n)) \neq 0.$$

This means there exists $i < n$ such that $B_i > B_n - t$ and $\zeta_i = \zeta_n$, which contradicts to our assumption.

Finally, since $A_n \geq A_i$ for $i < n$, after we apply

$$\text{Jac}_{\zeta_n(B_n-1), \dots, -\zeta_n(A_n+1)} \circ \cdots \circ \text{Jac}_{\zeta_n(B_n-t), \dots, -\zeta_n(A_n+t)}$$

to the full induced representation $\tau \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$, we will get $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$. So

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + t, B_n - t, l_n + 1, \eta_n, \zeta_n)) \hookrightarrow \tau \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$$

as the unique irreducible subrepresentation, and

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \circ_{i \in [1, t]} \text{Jac}_{\zeta_n(B_n-i), \dots, -\zeta_n(A_n+i)} \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + t, B_n - t, l_n + t, \eta_n, \zeta_n)).$$

So we have finished the proof. \square

6.3. Change sign. We choose an admissible order $>_\psi$, and we also fix a self-dual unitary irreducible supercuspidal representation ρ of $GL(d_\rho)$. We index the Jordan blocks in $Jord_\rho(\psi)$ such that

$$(\rho, A_{i+1}, B_{i+1}, \zeta_{i+1}) >_\psi (\rho, A_i, B_i, \zeta_i).$$

Suppose there exists n such that for $i > n$,

$$(\rho, A_i, B_i, \zeta_i) \gg \cup_{j=1}^n \{(\rho, A_j, B_j, \zeta_j)\}.$$

Moreover

$$\text{for } 1 < i \leq n, A_1 \geq A_i, B_1 = 1/2 \text{ or } 0, \text{ and } \zeta_i \neq \zeta_1.$$

We define ψ_- by

$$Jord(\psi_-) = Jord(\psi) \setminus \{(\rho, A_1, B_1, \zeta_1)\}.$$

We denote the restriction of $(\underline{l}, \underline{\eta})$ to $Jord(\psi_-)$ by $(\underline{l}_-, \underline{\eta}_-)$.

6.3.1. $B_1 = 0$.

Proposition 6.5. *For any $(\underline{l}, \underline{\eta})$, $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if*

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_1, 0, l_1, \eta_1, -\zeta_1)) \neq 0.$$

Moreover,

$$(6.4) \quad \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M, >_\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_1, 0, l_1, \eta_1, -\zeta_1)).$$

Proof. Let ψ_{\gg} dominates ψ with discrete diagonal restriction such that $T_1 = 0$. It suffices to prove (6.4) for ψ_{\gg} . When $l_1 = 0$, (6.4) is clear. So we can further assume $l_1 \neq 0$. Let $\psi_{\gg, -}$ be obtained from ψ_{\gg} by removing $(\rho, A_1, B_1, \zeta_1)$. Then

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1, 0, l_1, \eta_1, -\zeta_1)) &\hookrightarrow \langle 0, \dots, \zeta_1 A_1 \rangle \times \langle -\zeta_1 1, \dots, -\zeta_1 (A_1 - 1) \rangle \\ &\rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 2, 0, l_1 - 1, \eta_1, -\zeta_1)). \end{aligned}$$

as the unique irreducible subrepresentation. On the other hand,

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \langle 0, \dots, -\zeta_1 A_1 \rangle \times \langle \zeta_1 1, \dots, \zeta_1 (A_1 - 1) \rangle \\ &\rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 2, 0, l_1 - 1, \eta_1, \zeta_1)) \end{aligned}$$

$$\begin{aligned}
& \hookrightarrow \rho \times \langle -\zeta_1 1 \cdots, -\zeta_1 A_1 \rangle \times \langle \zeta_1 1, \dots, \zeta_1 (A_1 - 1) \rangle \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 2, 0, l_1 - 1, \eta_1, \zeta_1)) \\
& \cong \rho \times \langle \zeta_1 1, \dots, \zeta_1 (A_1 - 1) \rangle \times \langle -\zeta_1 1 \cdots, -\zeta_1 A_1 \rangle \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 2, 0, l_1 - 1, \eta_1, \zeta_1)) \\
& \hookrightarrow \rho \times \langle \zeta_1 1, \dots, \zeta_1 (A_1 - 1) \rangle \times \langle -\zeta_1 1 \cdots, -\zeta_1 (A_1 - 1) \rangle \times \rho^{|\zeta_1 A_1|} \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 2, 0, l_1 - 1, \eta_1, \zeta_1)).
\end{aligned}$$

Since $\rho^{|\zeta_1 A_1|} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 2, 0, l_1 - 1, \eta_1, \zeta_1))$ is irreducible, we have

$$\begin{aligned}
& \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-) \hookrightarrow \rho \times \langle \zeta_1 1, \dots, \zeta_1 (A_1 - 1) \rangle \times \langle -\zeta_1 1 \cdots, -\zeta_1 (A_1 - 1) \rangle \times \rho^{|\zeta_1 A_1|} \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 2, 0, l_1 - 1, \eta_1, \zeta_1)) \\
& \cong \rho \times \langle \zeta_1 1, \dots, \zeta_1 (A_1 - 1) \rangle \times \rho^{|\zeta_1 A_1|} \times \langle -\zeta_1 1 \cdots, -\zeta_1 (A_1 - 1) \rangle \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 2, 0, l_1 - 1, \eta_1, \zeta_1)).
\end{aligned}$$

Since $\text{Jac}_x \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-) = 0$ for $x = \zeta_1 1, \zeta_1 A_1$, then

$$\begin{aligned}
& \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-) \hookrightarrow \langle 0, \dots, \zeta_1 A_1 \rangle \times \langle -\zeta_1 1 \cdots, -\zeta_1 (A_1 - 1) \rangle \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 2, 0, l_1 - 1, \eta_1, \zeta_1)).
\end{aligned}$$

By induction on l_1 , we can assume

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 2, 0, l_1 - 1, \eta_1, \zeta_1)) = \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 2, 0, l_1 - 1, \eta_1, -\zeta_1)).$$

Then we necessarily have

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-) = \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg, -}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1, 0, l_1, \eta_1, -\zeta_1)).$$

This finishes the proof. □

6.3.2. $B_1 = 1/2$.

Proposition 6.6. *For any $(\underline{l}, \underline{\eta})$, one can construct*

$$\pi_{M, > \psi}^{\Sigma_0}(\psi^*, \underline{l}^*, \underline{\eta}^*) := \begin{cases} \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 + 1, 1/2, l_1 + 1, -\eta_1, -\zeta_1)) & \text{if } \eta_1 = +1, \\ \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 + 1, 1/2, l_1, -\eta_1, -\zeta_1)) & \text{if } \eta_1 = -1. \end{cases}$$

In case $l_1 = (A_1 + \frac{1}{2})/2$, we fix $\eta_1 = -1$. Then

$$\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0 \text{ if and only if } \pi_{M, > \psi}^{\Sigma_0}(\psi^*, \underline{l}^*, \underline{\eta}^*) \neq 0.$$

Moreover,

$$\pi_{M, > \psi}^{\Sigma_0}(\psi^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, \dots, -\zeta_1 (A_1 + 1) \rangle \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$$

as the unique irreducible subrepresentation, and

$$\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \text{Jac}_{-\zeta_1 1/2, \dots, -\zeta_1 (A_1 + 1)} \pi_{M, > \psi}^{\Sigma_0}(\psi^*, \underline{l}^*, \underline{\eta}^*).$$

Proof. Let us choose ψ_{\gg} dominating ψ with discrete diagonal restriction, and we require $T_1 = 0$. Then it determines ψ_{\gg}^* which dominates ψ^* . We will assume the proposition for ψ_{\gg} . Then

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, \dots, -\zeta_1(A_1 + 1) \rangle \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}).$$

Suppose $\pi_{M, > \psi}^{\Sigma_0}(\psi^*, \underline{l}^*, \underline{\eta}^*) \neq 0$, then after we apply

$$\circ_{i>1} \text{Jac}_{(\rho, A_i+T_i, B_i+T_i, \zeta_i) \mapsto (\rho, A_i, B_i, \zeta_i)}$$

(i decreases) and Jac_{X^c} to the full induced representation

$$(6.5) \quad \langle -\zeta_1 1/2, \dots, -\zeta_1(A_1 + 1) \rangle \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta})$$

we should get something nonzero. Since

$$-\zeta_1 1/2 \text{ and } \zeta_1(A_1 + 1) \notin [\zeta_i(A_i + T_i), \zeta_i(B_i + 1)]$$

for $i > 1$, $\circ_{i>1} \text{Jac}_{(\rho, A_i+T_i, B_i+T_i, \zeta_i) \mapsto (\rho, A_i, B_i, \zeta_i)}$ and Jac_{X^c} can only apply to $\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta})$. Therefore

$$\circ_{i>1} \text{Jac}_{(\rho, A_i+T_i, B_i+T_i, \zeta_i) \mapsto (\rho, A_i, B_i, \zeta_i)} \circ \text{Jac}_{X^c}(6.5) = \langle -\zeta_1 1/2, \dots, -\zeta_1(A_1 + 1) \rangle \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0.$$

This shows $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$.

Suppose $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, let

$$\mathcal{C}_{X_i} := \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix},$$

then

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, \dots, -\zeta_1(A_1 + 1) \rangle \times (\times_{i>1} \mathcal{C}_{X_i}) \times \mathcal{C}_{X^c} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}),$$

where i increases. For $i > n$, we have $B_i > A_1 + 1$, so \mathcal{C}_{X_i} and $\langle -\zeta_1 1/2, \dots, -\zeta_1(A_1 + 1) \rangle$ are interchangeable. For $i < n$, we have $A_1 \geq A_i$ and $\zeta_i = -\zeta_1$, so

$$[\zeta_i(B_i + 1), \zeta_i(A_i + 1)] \subseteq [-\zeta_1 1/2, -\zeta_1(A_1 + 1)].$$

It follows \mathcal{C}_{X_i} and $\langle -\zeta_1 1/2, \dots, -\zeta_1(A_1 + 1) \rangle$ are also interchangeable. Therefore,

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow (\times_{i>1} \mathcal{C}_{X_i}) \times \mathcal{C}_{X^c} \times \langle -\zeta_1 1/2, \dots, -\zeta_1(A_1 + 1) \rangle \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}).$$

This implies $\pi_{M, > \psi}^{\Sigma_0}(\psi^*, \underline{l}^*, \underline{\eta}^*) \neq 0$, and

$$\pi_{M, > \psi}^{\Sigma_0}(\psi^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, \dots, -\zeta_1(A_1 + 1) \rangle \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}).$$

To see $\pi_{M, > \psi}^{\Sigma_0}(\psi^*, \underline{l}^*, \underline{\eta}^*)$ is the unique irreducible subrepresentation, it suffices to check that

$$\text{Jac}_{-\zeta_1 1/2, \dots, -\zeta_1(A_1+1)} \left(\langle -\zeta_1 1/2, \dots, -\zeta_1(A_1 + 1) \rangle \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \right) = \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}).$$

As a consequence, we also get

$$\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \text{Jac}_{-\zeta_1 1/2, \dots, -\zeta_1(A_1+1)} \pi_{M, > \psi}^{\Sigma_0}(\psi^*, \underline{l}^*, \underline{\eta}^*).$$

To complete the proof, we still need to show the proposition for ψ_{\gg} , and we leave it to the next lemma. \square

Lemma 6.7. *Proposition 6.6 holds for ψ_{\gg} .*

Proof. It is clear that $\pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \neq 0$ and $\pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) \neq 0$ in this case. So we only need to show

$$\pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, \dots, -\zeta_1(A_1 + 1) \rangle \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta})$$

as the unique irreducible subrepresentation, and

$$\pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) = \text{Jac}_{-\zeta_1 1/2, \dots, -\zeta_1(A_1+1)} \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*).$$

Let $\psi_{\gg,-}$ be obtained from ψ_{\gg} by removing $(\rho, A_1, B_1, \zeta_1)$. When $l_1 = 0$ and $\eta_1 = -1$, the lemma is clear. When $A_1 = 1/2$, then necessarily $l_1 = 0$. In this case, if $\eta = +1$, then

$$\begin{aligned} \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) &\hookrightarrow \langle -\zeta_1 1/2, \dots, \zeta_1 3/2 \rangle \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-) \\ &\hookrightarrow \rho ||^{-\zeta_1 1/2} \times \rho ||^{\zeta_1 1/2} \times ||^{\zeta_1 3/2} \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-) \\ &\cong \rho ||^{-\zeta_1 1/2} \times \rho ||^{\zeta_1 1/2} \times ||^{-\zeta_1 3/2} \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-) \\ &\cong \rho ||^{-\zeta_1 1/2} \times ||^{-\zeta_1 3/2} \times \rho ||^{\zeta_1 1/2} \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-). \end{aligned}$$

There exists an irreducible constituent σ of $\rho ||^{\zeta_1 1/2} \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-)$ such that

$$\pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \rho ||^{-\zeta_1 1/2} \times ||^{-\zeta_1 3/2} \times \sigma.$$

Since $\text{Jac}_{-\zeta_1 3/2} \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) = 0$, we must have

$$\pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, -\zeta_1 3/2 \rangle \times \sigma.$$

Suppose $\text{Jac}_{-\zeta_1 1/2} \sigma \neq 0$, then there exists an irreducible constituent σ' of $\text{Jac}_{-\zeta_1 1/2} \sigma$ such that

$$\begin{aligned} \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) &\hookrightarrow \langle -\zeta_1 1/2, -\zeta_1 3/2 \rangle \times \rho ||^{-\zeta_1 1/2} \rtimes \sigma' \\ &\cong \rho ||^{-\zeta_1 1/2} \times \langle -\zeta_1 1/2, -\zeta_1 3/2 \rangle \rtimes \sigma'. \end{aligned}$$

This implies $\text{Jac}_{-\zeta_1 1/2, -\zeta_1 1/2} \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) \neq 0$, which is impossible. Therefore, we must have $\text{Jac}_{\zeta_1 1/2} \sigma \neq 0$. In particular, this means $\sigma = \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta})$. So

$$\pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, -\zeta_1 3/2 \rangle \times \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}).$$

To see $\pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*)$ is the unique irreducible subrepresentation, it suffices to check that

$$\text{Jac}_{-\zeta_1 1/2, -\zeta_1 3/2} \left(\langle -\zeta_1 1/2, -\zeta_1 3/2 \rangle \times \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \right) = \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}).$$

As a consequence,

$$\pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) = \text{Jac}_{-\zeta_1 1/2, -\zeta_1 3/2} \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*).$$

Next we would like to prove this lemma by induction on A_1 . Let $A_1 > 1/2$. Suppose $\eta_1 = +1$, then

$$\begin{aligned} \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) &\hookrightarrow \langle -\zeta_1 1/2, \dots, \zeta_1(A_1 + 1) \rangle \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1, 3/2, l_1, -\eta_1, -\zeta_1)) \\ &\hookrightarrow \langle -\zeta_1 1/2, \dots, \zeta_1(A_1 + 1) \rangle \times \langle -\zeta_1 3/2, \dots, -\zeta_1 A_1 \rangle \\ &\rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 1, 1/2, l_1, -\eta_1, -\zeta_1)) \\ &\hookrightarrow \rho ||^{-\zeta_1 1/2} \times \langle -\zeta_1 3/2, \dots, -\zeta_1 A_1 \rangle \times \langle \zeta_1 1/2, \dots, \zeta_1 A_1 \rangle \times \rho ||^{\zeta_1(A_1+1)} \\ &\rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 1, 1/2, l_1, -\eta_1, -\zeta_1)) \\ &\cong \rho ||^{-\zeta_1 1/2} \times \langle -\zeta_1 3/2, \dots, -\zeta_1 A_1 \rangle \times \langle \zeta_1 1/2, \dots, \zeta_1 A_1 \rangle \times \rho ||^{-\zeta_1(A_1+1)} \\ &\rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 1, 1/2, l_1, -\eta_1, -\zeta_1)) \\ &\cong \rho ||^{-\zeta_1 1/2} \times \langle -\zeta_1 3/2, \dots, -\zeta_1 A_1 \rangle \times \rho ||^{-\zeta_1(A_1+1)} \times \langle \zeta_1 1/2, \dots, \zeta_1 A_1 \rangle \end{aligned}$$

$$\rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, -, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 1, 1/2, l_1, -\eta_1, -\zeta_1))$$

There exists an irreducible constituent σ of

$$< \zeta_1 1/2, \dots, \zeta_1 A_1 > \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, -, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 1, 1/2, l_1, -\eta_1, -\zeta_1))$$

such that

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \rho ||^{-\zeta_1 1/2} \times < -\zeta_1 3/2, \dots, -\zeta_1 A_1 > \times \rho ||^{-\zeta_1 (A_1 + 1)} \rtimes \sigma$$

Since $\text{Jac}_x \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) = 0$ for $x \in [-\zeta_1 3/2, -\zeta_1 (A_1 + 1)]$, then

$$(6.6) \quad \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow < -\zeta_1 1/2, \dots, -\zeta_1 (A_1 + 1) > \rtimes \sigma.$$

If we apply $\text{Jac}_{-\zeta_1 1/2, \dots, -\zeta_1 (A_1 + 1)}$ to the full induced representation in (6.6), then $\text{Jac}_{-\zeta_1 (A_1 + 1)}$ can only apply to $< -\zeta_1 1/2, \dots, -\zeta_1 (A_1 + 1) >$. As a consequence, we must have the whole Jacquet functor $\text{Jac}_{-\zeta_1 1/2, \dots, -\zeta_1 (A_1 + 1)}$ applied to $< -\zeta_1 1/2, \dots, -\zeta_1 (A_1 + 1) >$, and hence

$$\text{Jac}_{-\zeta_1 1/2, \dots, -\zeta_1 (A_1 + 1)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) = \sigma,$$

which is irreducible. Therefore

$$(6.7) \quad \sigma \hookrightarrow < \zeta_1 1/2, \dots, \zeta_1 A_1 > \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, -, \underline{l}_-, \underline{\eta}_-; (\rho, A_1 - 1, 1/2, l_1, -\eta_1, -\zeta_1)).$$

By induction, $\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta})$ is the unique irreducible subrepresentation of the induced representation in (6.7), so it has to be equal to σ . Hence

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow < -\zeta_1 1/2, \dots, -\zeta_1 (A_1 + 1) > \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}).$$

To see $\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*)$ is the unique irreducible subrepresentation, it suffices to check that

$$\text{Jac}_{-\zeta_1 1/2, \dots, -\zeta_1 (A_1 + 1)} \left(< -\zeta_1 1/2, \dots, -\zeta_1 (A_1 + 1) > \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \right) = \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}).$$

As a consequence,

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) = \text{Jac}_{-\zeta_1 1/2, \dots, -\zeta_1 (A_1 + 1)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}^*, \underline{l}^*, \underline{\eta}^*).$$

Suppose $\eta_1 = -1$, we can also assume $l_1 \neq 0$, then proof is the same. □

7. GENERAL PROCEDURE

The three operations (“Pull”, “Expand”, and “Change sign”) introduced in the previous section allow us to develop a procedure to find the combinatorial conditions for the nonvanishing of $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$.

7.1. Step one. We choose an admissible order $>_\psi$, and we also fix a self-dual unitary irreducible supercuspidal representation ρ of $GL(d_\rho)$. We index the Jordan blocks in $Jord_\rho(\psi)$ such that

$$(\rho, A_i, B_i, \zeta_i) >_\psi (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

We choose n such that for $i > n$,

$$(\rho, A_i, B_i, \zeta_i) \gg_2 \cup_{j=1}^n \{(\rho, A_j, B_j, \zeta_j)\},$$

and the Jordan blocks for $i > n$ are in “good shape” (see Remark 4.4). Then for $i \leq n$, let us choose (ρ, A, B, ζ) so that A is maximal. We consider the set

$$(7.1) \quad \{(\rho, A_i, B_i, \zeta_i) \text{ for } i \leq n : [A_i, B_i] \subsetneq [A, B] \text{ and } \zeta_i = \zeta\}.$$

If this set is nonempty, we take (ρ, A', B', ζ') such that A' is maximal within the set. We can rearrange the order $>_\psi$ for $i \leq n$, so that

$$(\rho, A_n, B_n, \zeta_n) = (\rho, A, B, \zeta) \text{ and } (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1}) = (\rho, A', B', \zeta').$$

Then we can “Pull” the pairs $(\rho, A_n, B_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1})$ using Proposition 6.1. Suppose the set (7.1) is empty, but there exists (ρ, A', B', ζ') such that

$$[A', B'] = [A, B] \text{ and } \zeta' = \zeta,$$

then we can again rearrange the order $>_\psi$ for $i \leq n$, so that

$$(\rho, A_n, B_n, \zeta_n) = (\rho, A, B, \zeta) \text{ and } (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1}) = (\rho, A', B', \zeta').$$

And we can “Pull” the pairs $(\rho, A_n, B_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1})$ using Proposition 6.3.

7.2. Step two. Following Step one, we suppose the set

$$(7.2) \quad \{(\rho, A_i, B_i, \zeta_i) \text{ for } i \leq n : [A_i, B_i] \subseteq [A, B] \text{ and } \zeta_i = \zeta\} \setminus \{(\rho, A, B, \zeta)\}$$

is empty. We can still rearrange the order $>_\psi$ for $i \leq n$ such that

$$(\rho, A_n, B_n, \zeta_n) = (\rho, A, B, \zeta).$$

Then we “Expand” $[A_n, B_n]$, and use Proposition 6.4.

7.3. Step three. Following Step two, let us denote the “Expansion” of $[A_n, B_n]$ by $[A_n^*, B_n^*]$. The set (7.1) becomes

$$\{(\rho, A_i, B_i, \zeta_i) \text{ for } i < n : [A_i, B_i] \subsetneq [A_n^*, B_n^*] \text{ and } \zeta_i = \zeta_n\}.$$

If this set is nonempty, then we are back to Step one. If this set is empty, then by our definition of “Expand”, it is necessary that $B_n^* = 1/2$ or 0, and $\zeta_i \neq \zeta_n$ for all $i < n$. In this case, we can change the order for $i \leq n$ again by switching $(\rho, A_n^*, B_n^*, \zeta_n)$ with $(\rho, A_i, B_i, \zeta_i)$ one by one as i goes from $n-1$ to 1. Then we can “Change sign” of $(\rho, A_n^*, B_n^*, \zeta_n)$, and use Proposition 6.5 or Proposition 6.6. After that, we are back to Step one again.

7.4. Step four. By the above three steps, we will end up with a collection of parameters $\{\psi^*\}$ such that $Jord_\rho(\psi^*)$ is in “good shape” (cf. Proposition 6.1 and Proposition 6.3). Then we can change ρ and repeat all the previous steps to $\{\psi^*\}$.

8. EXAMPLE

In this section, we want to demonstrate how our procedure (cf. Section 7) works in a simple example. We fix ρ and choose $\psi \in \bar{\Psi}(G)$, such that

$$Jord(\psi) = \{(\rho, A_3, B_3, \zeta_3), (\rho, A_2, B_2, \zeta_2), (\rho, A_1, B_1, \zeta_1)\}.$$

We also assume

- $A_i, B_i \in \mathbb{Z}$ for $i = 1, 2, 3$;
- $A_3 \geq A_2 \geq A_1$ and $B_3 \geq B_2 \geq B_1$;
- $\zeta_3 = \zeta_1 = +1$ and $\zeta_2 = -1$.

To visualize $Jord(\psi)$, we will represent the Jordan blocks $(\rho, A_i, B_i, \zeta_i)$ by arcs with end points A_i, B_i on the real line. If $\zeta_i = +1$ (resp. -1), we will draw it black (resp. red). Here is a rough picture of $Jord(\psi)$:



FIGURE 1. ψ

We put an order $>_\psi$ on $Jord(\psi)$ such that

$$(\rho, A_3, B_3, \zeta_3) >_\psi (\rho, A_2, B_2, \zeta_2) >_\psi (\rho, A_1, B_1, \zeta_1).$$

We would like to find all $(\underline{l}, \underline{\eta})$ such that $\pi_{>_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$.

First, we “Expand” $[A_3, B_3]$ to $[A_3^*, B_3^*]$ such that $B_3^* = B_1$, and we denote the resulting parameter by ψ^* .

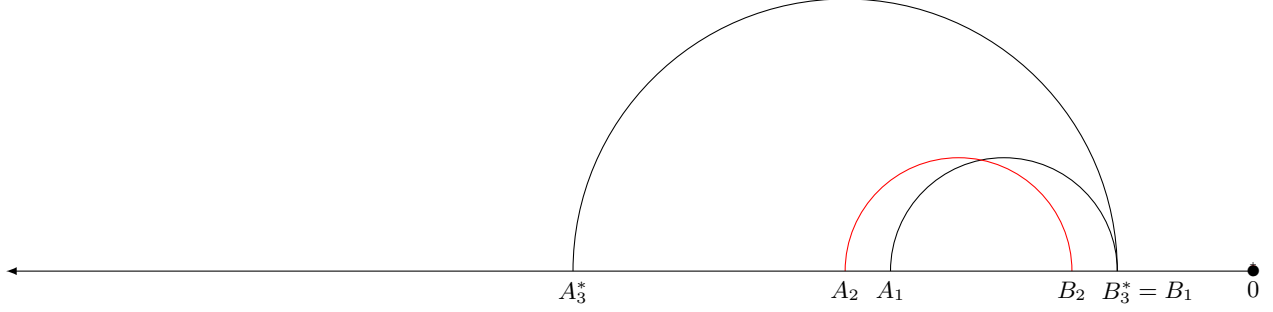


FIGURE 2. ψ^*

Then $\pi_{>_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{>_\psi}^{\Sigma_0}(\psi^*, \underline{l}^*, \underline{\eta}^*)$, where

$$l_1^* = l_1, \quad l_2^* = l_2, \quad l_3^* = l_3 + (B_3 - B_1),$$

and

$$\eta_1^* = \eta_1, \quad \eta_2^* = \eta_2, \quad \eta_3^* = \eta_3.$$

Next, we change the order $>_\psi$ to $>'_\psi$:

$$(\rho, A_3, B_3, \zeta_3) >'_\psi (\rho, A_1, B_1, \zeta_1) >'_\psi (\rho, A_2, B_2, \zeta_2).$$

So $\pi_{>_\psi}^{\Sigma_0}(\psi^*, \underline{l}^*, \underline{\eta}^*) = \pi_{>'_\psi}^{\Sigma_0}(\psi^*, \underline{l}'^*, \underline{\eta}'^*)$, where

$$l_1'^* = l_1^*, \quad l_2'^* = l_2^*, \quad l_3'^* = l_3^*,$$

and

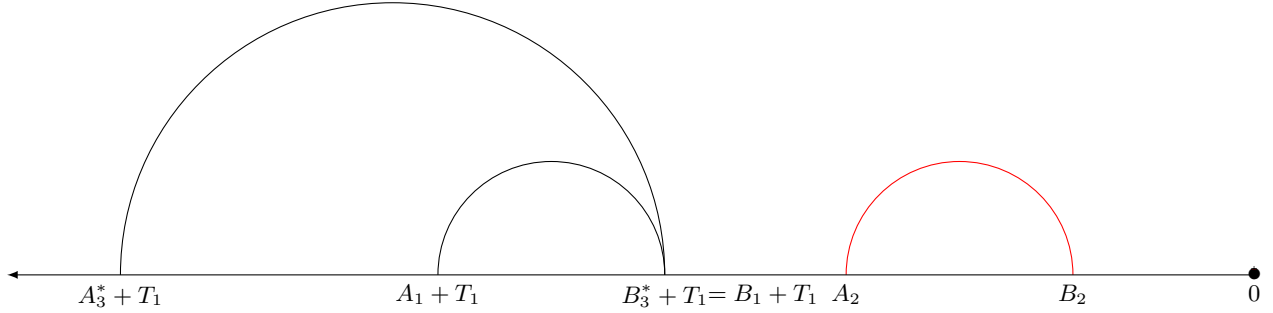
$$\eta_1'^* = (-1)^{A_2 - B_2 + 1} \eta_1^*, \quad \eta_2'^* = (-1)^{A_1 - B_1 + 1} \eta_2^*, \quad \eta_3'^* = \eta_3^*.$$

Then we can “Pull” $[A_3^*, B_3^*]$, $[A_1, B_1]$. As a consequence, we have $\pi_{>'_\psi}^{\Sigma_0}(\psi^*, \underline{l}'^*, \underline{\eta}'^*) \neq 0$ if and only if all the three conditions below are satisfied.

- (1) $\pi_{>'_\psi}^{\Sigma_0}(\psi_1^*, \underline{l}'^*, \underline{\eta}'^*) \neq 0$,
- (2) $\pi_{>'_\psi}^{\Sigma_0}(\psi_2^*, \underline{l}'^*, \underline{\eta}'^*) \neq 0$,
- (3) $\pi_{>''_\psi}^{\Sigma_0}(\psi_3^*, \underline{l}''^*, \underline{\eta}''^*) \neq 0$.

From each of these cases, we will get some explicit conditions on $(\underline{l}, \underline{\eta})$.

Case (1): ψ_1^* is obtained from ψ^* by shifting both $[A_3^*, B_3^*]$ and $[A_1, B_1]$ away by T_1 , so that $(\rho, A_1 + T_1, B_1 + T_1, \zeta_1) \gg (\rho, A_2, B_2, \zeta_2)$.

FIGURE 3. ψ_1^*

Since $Jord(\psi_1^*)$ is in “good shape”, we have $\pi_{>\psi}^{\Sigma_0}(\psi_1^*, \underline{l}^*, \underline{\eta}^*) \neq 0$ if and only if

$$(8.1) \quad \begin{cases} \eta_3^* = (-1)^{A_1-B_1} \eta_1^* & \Rightarrow 0 \leq l_3^* - l_1^* \leq (A_3^* - B_3^*) - (A_1 - B_1), \\ \eta_3^* \neq (-1)^{A_1-B_1} \eta_1^* & \Rightarrow l_3^* + l_1^* > A_1 - B_1. \end{cases}$$

Now we want to translate these conditions to that on $(\underline{l}, \underline{\eta})$. Note

$$\eta_3^* = \eta_3 = \eta_3, \quad \eta_1^* = (-1)^{A_2-B_2+1} \eta_1 = (-1)^{A_2-B_2+1} \eta_1,$$

and

$$l_3^* = l_3^* = l_3 + (B_3 - B_1), \quad l_1^* = l_1^* = l_1.$$

So we will get the following conditions from (8.1).

- If $\eta_3 \neq (-1)^{(A_1-B_1)+(A_2-B_2)} \eta_1$, then

$$0 \leq l_3 + (B_3 - B_1) - l_1 \leq (A_3 + (B_3 - B_1) - B_1) - (A_1 - B_1),$$

which implies

$$-(B_3 - B_1) \leq l_3 - l_1 \leq A_3 - A_1.$$

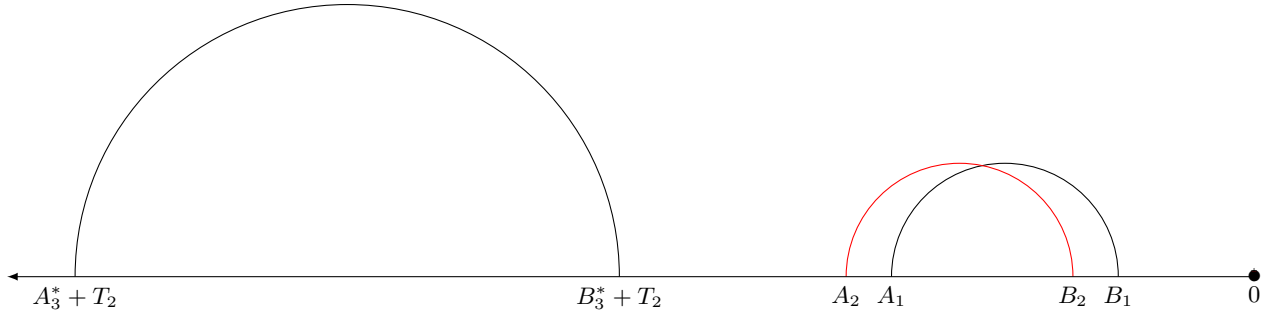
- If $\eta_3 = (-1)^{(A_1-B_1)+(A_2-B_2)} \eta_1$, then

$$l_3 + (B_3 - B_1) + l_1 > A_1 - B_1,$$

which implies

$$l_3 + l_1 > A_1 - B_3.$$

Case (2): ψ_2^* is obtained from ψ^* by shifting $[A_3^*, B_3^*]$ away by T_2 , so that $(\rho, A_3^* + T_2, B_3^* + T_2, \zeta_3) \gg \{(\rho, A_2, B_2, \zeta_2), (\rho, A_1, B_1, \zeta_1)\}$.

FIGURE 4. ψ_2^*

Note $\pi_{>\psi}^{\Sigma_0}(\psi_2^*, \underline{l}^*, \underline{\eta}^*) = \pi_{>\psi}^{\Sigma_0}(\psi_2^*, \underline{l}^*, \underline{\eta}^*)$. So we can “Expand” $[A_2, B_2]$ to $[A_2^*, B_2^*]$ such that $B_2^* = 0$, and we denote the resulting parameter by ψ_2^{**} .

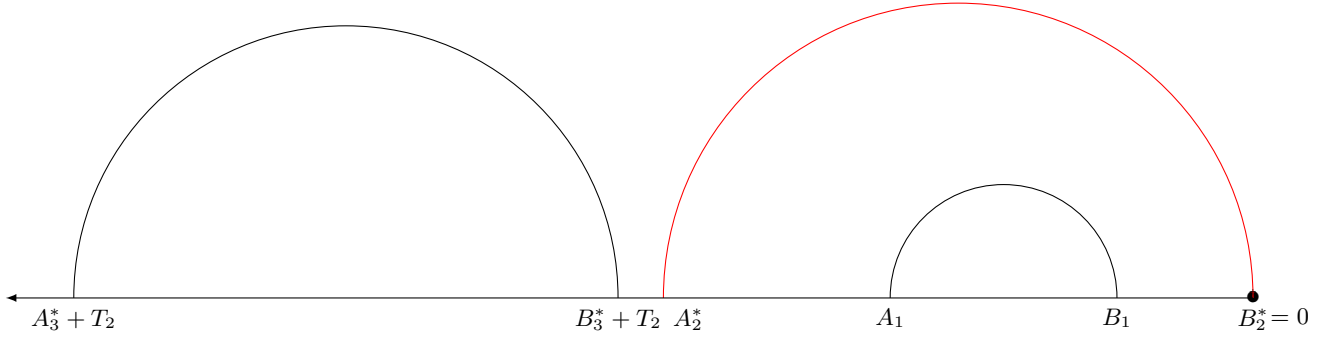


FIGURE 5. ψ_2^{**}

Then $\pi_{>\psi}^{\Sigma_0}(\psi_2^*, \underline{l}^*, \underline{\eta}^*) = \pi_{>\psi}^{\Sigma_0}(\psi_2^{**}, \underline{l}^{**}, \underline{\eta}^{**})$, where

$$l_1^{**} = l_1^*, \quad l_2^{**} = l_2^* + B_2, \quad l_3^{**} = l_3^*,$$

and

$$\eta_1^{**} = \eta_1^*, \quad \eta_2^{**} = \eta_2^*, \quad \eta_3^{**} = \eta_3^*.$$

Finally, we can change the order again

$$\pi_{>\psi}^{\Sigma_0}(\psi_2^{**}, \underline{l}^{**}, \underline{\eta}^{**}) = \pi_{>\psi}^{\Sigma_0}(\psi_2^{**}, \underline{l}'^{**}, \underline{\eta}'^{**}),$$

where

$$l_1'^{**} = l_1^{**}, \quad l_2'^{**} = l_2^{**}, \quad l_3'^{**} = l_3^{**},$$

and

$$\eta_1'^{**} = (-1)^{A_2 - B_2 + 1} \eta_1^{**}, \quad \eta_2'^{**} = (-1)^{A_1 - B_1 + 1} \eta_2^{**}, \quad \eta_3'^{**} = \eta_3^{**}.$$

Then

$$\pi_{>\psi}^{\Sigma_0}(\psi_2^{**}, \underline{l}'^{**}, \underline{\eta}'^{**}) = \pi_{>\psi}^{\Sigma_0}(\psi_2^{**}, \underline{l}'^{**}, \underline{\eta}'^{**}; (\rho, A_2^*, B_2^*, \eta_2'^{**}, -\zeta_2)) \neq 0,$$

if and only if

$$(8.2) \quad \begin{cases} \eta_1'^{**} = (-1)^{A_2 - B_2} \eta_2'^{**} & \Rightarrow 0 \leq l_2'^{**} - l_1'^{**} \leq (A_2^* - B_2^*) - (A_1 - B_1), \\ \eta_1'^{**} \neq (-1)^{A_2 - B_2} \eta_2'^{**} & \Rightarrow l_2'^{**} + l_1'^{**} > A_1 - B_1. \end{cases}$$

Now we want to translate these conditions to that on $(\underline{l}, \underline{\eta})$. Note

$$\begin{aligned} \eta_1'^{**} &= (-1)^{A_2 - B_2 + 1} \eta_1^{**} = (-1)^{A_2 - B_2 + 1} \eta_1^* = (-1)^{A_2 - B_2 + 1} \eta_1 \\ \eta_2'^{**} &= (-1)^{A_1 - B_1 + 1} \eta_2^{**} = (-1)^{A_1 - B_1 + 1} \eta_2^* = (-1)^{A_1 - B_1 + 1} \eta_2 \end{aligned}$$

and

$$l_2'^{**} = l_2^{**} = l_2^* + B_2 = l_2 + B_2, \quad l_1'^{**} = l_1^{**} = l_1^* = l_1.$$

So we will get the following conditions from (8.2).

- If $\eta_2 = (-1)^{A_1 - B_1} \eta_1$, then

$$0 \leq l_2 + B_2 - l_1 \leq (A_2 + B_2 - 0) - (A_1 - B_1),$$

which implies

$$-B_2 \leq l_2 - l_1 \leq A_2 - (A_1 - B_1).$$

- If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, then

$$l_2 + B_2 + l_1 > A_1 - B_1,$$

which implies

$$l_2 + l_1 > (A_1 - B_1) - B_2.$$

Case (3): ψ_3^* is obtained from ψ^* by shifting $[A_1, B_1]$ away by T_3 , so that $(\rho, A_1 + T_3, B_1 + T_3, \zeta_1) \gg \{(\rho, A_3^*, B_3^*, \zeta_3), (\rho, A_2, B_2, \zeta_2)\}$.

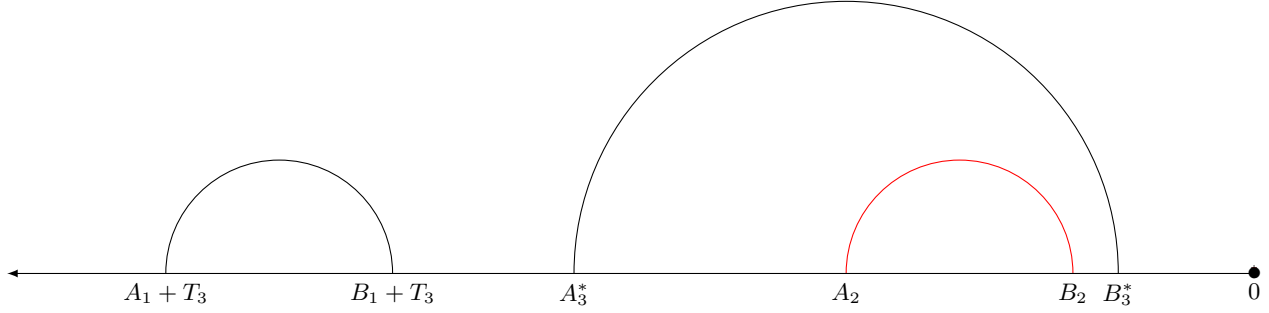


FIGURE 6. ψ_3^*

The order $>''_\psi$ is given by

$$(\rho, A_1, B_1, \zeta_1) >''_\psi (\rho, A_3, B_3, \zeta_3) >''_\psi (\rho, A_2, B_2, \zeta_2).$$

And $\pi_{>''_\psi}^{\Sigma_0}(\psi^*, \underline{l}'^*, \underline{\eta}'^*) = \pi_{>''_\psi}^{\Sigma_0}(\psi^*, \underline{l}''^*, \underline{\eta}''^*)$, where $(\underline{l}''^*, \underline{\eta}''^*) = S^+(\underline{l}'^*, \underline{\eta}'^*)$. In particular,

$$\eta_2''^* = \eta_2'^*, \quad l_2''^* = l_2'^*.$$

Then we can “Expand” $[A_3^*, B_3^*]$ to $[A_3^{**}, B_3^{**}]$ such that $B_3^{**} = 0$, and we denote the resulting parameter by ψ_3^{**} .

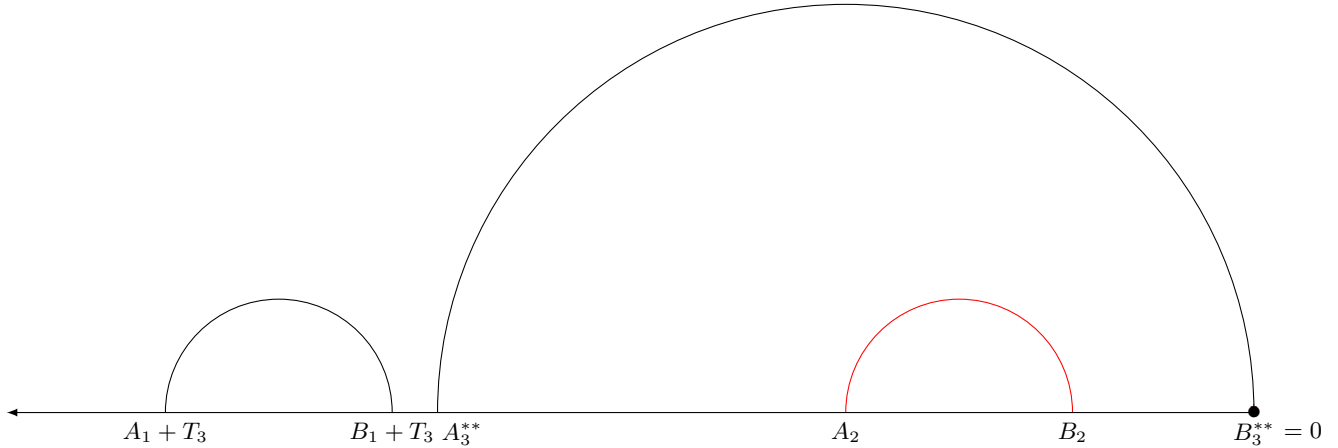


FIGURE 7. ψ_3^{**}

It follows $\pi_{>''_\psi}^{\Sigma_0}(\psi_3^*, \underline{l}''^*, \underline{\eta}''^*) = \pi_{>''_\psi}^{\Sigma_0}(\psi_3^*, \underline{l}''^{**}, \underline{\eta}''^{**})$, where

$$l_1''^{**} = l_1''^*, \quad l_2''^{**} = l_2''^*, \quad l_3''^{**} = l_3''^* + B_1,$$

and

$$\eta_1''^{**} = \eta_1''^*, \quad \eta_2''^{**} = \eta_2''^*, \quad \eta_3''^{**} = \eta_3''^*.$$

We change the order $>''_\psi$ to $>'''_\psi$:

$$(\rho, A_1, B_1, \zeta_1) >'''_\psi (\rho, A_2, B_2, \zeta_2) >'''_\psi (\rho, A_3, B_3, \zeta_3),$$

then

$$\pi_{>'''_\psi}^{\Sigma_0}(\psi_2^{**}, \underline{l}''^{**}, \underline{\eta}''^{**}) = \pi_{>'''_\psi}^{\Sigma_0}(\psi_2^{**}, \underline{l}'''^{**}, \underline{\eta}'''^{**}),$$

where

$$l_1'''^{**} = l_1''^{**}, \quad l_2'''^{**} = l_2''^{**}, \quad l_3'''^{**} = l_3''^{**},$$

and

$$\eta_1'''^{**} = \eta_1''^{**}, \quad \eta_2'''^{**} = (-1)^{A_3-B_3+1} \eta_2''^{**}, \quad \eta_3'''^{**} = (-1)^{A_2-B_2+1} \eta_3''^{**}.$$

Then

$$\pi_{>'''_\psi}^{\Sigma_0}(\psi_3^{**}, \underline{l}'''^{**}, \underline{\eta}'''^{**}) = \pi_{>'''_\psi}^{\Sigma_0}(\psi_{3-}^{**}, \underline{l}_{-}'''^{**}, \underline{\eta}_{-}'''^{**}; (\rho, A_3^{**}, B_3^{**}, \eta_3'''^{**}, -\zeta_3)) \neq 0,$$

if and only if

$$(8.3) \quad \begin{cases} \eta_2'''^{**} = (-1)^{A_3-B_3} \eta_3'''^{**} & \Rightarrow 0 \leq l_3'''^{**} - l_2'''^{**} \leq (A_3^{**} - B_3^{**}) - (A_2 - B_2), \\ \eta_2'''^{**} \neq (-1)^{A_3-B_3} \eta_3'''^{**} & \Rightarrow l_3'''^{**} + l_2'''^{**} > A_2 - B_2. \end{cases}$$

Now we want to translate these conditions to that on $(\underline{l}, \underline{\eta})$. Note

$$\begin{aligned} \eta_2'''^{**} &= (-1)^{A_3-B_3+1} \eta_2''^{**} = (-1)^{A_3-B_3+1} \eta_2''^* = (-1)^{A_3-B_3+1} \eta_2'^* \\ &= (-1)^{A_3-B_3+1} (-1)^{A_1-B_1+1} \eta_2^* = (-1)^{(A_3-B_3)+(A_1-B_1)} \eta_2^* \\ \eta_3'''^{**} &= (-1)^{A_2-B_2+1} \eta_3''^{**} \end{aligned}$$

and

$$\begin{aligned} l_2'''^{**} &= l_2''^{**} = l_2''^* = l_2'^* = l_2^* = l_2 \\ l_3'''^{**} &= l_3''^{**} = l_3''^* + B_1 \end{aligned}$$

To proceed further, we need to use the formula for $(\underline{l}'^*, \underline{\eta}'^*) = S^+(\underline{l}^*, \underline{\eta}^*)$.

- If $\eta_3'^* \neq (-1)^{A_1-B_1} \eta_1'^*$, then $\eta_1''^* = (-1)^{A_3-B_3} \eta_3''^*$ and

$$\begin{cases} l_1'^* = l_1''^* \\ l_3'^* - l_3''^* = (A_1 - B_1 - 2l_1'^*) + 1 \\ \eta_1''^* = (-1)^{A_3-B_3} \eta_1'^* \end{cases}$$

It follows

$$\begin{aligned} \eta_3'^* &\neq (-1)^{A_1-B_1} \eta_1'^* \\ \Rightarrow \eta_3^* &\neq (-1)^{A_1-B_1} (-1)^{A_2-B_2+1} \eta_1^* \\ \Rightarrow \eta_3 &\neq (-1)^{(A_1-B_1)+(A_2-B_2)+1} \eta_1 \\ \Rightarrow \eta_3 &= (-1)^{(A_1-B_1)+(A_2-B_2)} \eta_1 \end{aligned}$$

We also have

$$\begin{aligned} (-1)^{A_3-B_3} \eta_3''^* &= \eta_1''^* = (-1)^{A_3-B_3} \eta_1'^* \\ \Rightarrow \eta_3''^* &= \eta_1'^* \\ \Rightarrow \eta_3''^* &= (-1)^{A_2-B_2+1} \eta_1^* \\ \Rightarrow \eta_3''^* &= (-1)^{A_2-B_2+1} \eta_1 \end{aligned}$$

and

$$\begin{aligned}
l_3''^* &= l_3'^* - (A_1 - B_1 - 2l_1'^*) - 1 \\
&= l_3^* - (A_1 - B_1 - 2l_1^*) - 1 \\
&= l_3 + (B_3 - B_1) - (A_1 - B_1 - 2l_1) - 1 \\
&= l_3 + B_3 - A_1 + 2l_1 - 1
\end{aligned}$$

So we will get the following conditions from (8.3).

– If $\eta_2 = (-1)^{A_1-B_1}\eta_1$, then

$$0 \leq (l_3 + B_3 - A_1 + 2l_1 - 1) + B_1 - l_2 \leq (A_3 + B_3 - 0) - (A_2 - B_2)$$

which implies

$$(A_1 - B_1) - B_3 + 1 \leq l_3 - l_2 + 2l_1 \leq A_3 + (A_1 - B_1) - (A_2 - B_2) + 1$$

– If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, then

$$(l_3 + B_3 - A_1 + 2l_1 - 1) + B_1 + l_2 > A_2 - B_2$$

which implies

$$l_3 + l_2 + 2l_1 > (A_1 - B_1) + (A_2 - B_2) - B_3 + 1$$

• If $\eta_3'^* = (-1)^{A_1-B_1}\eta_1'^*$ and

$$l_3'^* - l_1'^* < (A_3^* - B_3^*)/2 - (A_1 - B_1) + l_1'^*,$$

then $\eta_1''^* \neq (-1)^{A_3-B_3}\eta_3''^*$ and

$$\begin{cases} l_1'^* = l_1''^* \\ l_3''^* - l_3'^* = (A_1 - B_1 - 2l_1'^*) + 1 \\ \eta_1''^* = (-1)^{A_3-B_3}\eta_1'^* \end{cases}$$

It follows

$$\begin{aligned}
\eta_3'^* &= (-1)^{A_1-B_1}\eta_1'^* \\
\Rightarrow \eta_3^* &= (-1)^{A_1-B_1}(-1)^{A_2-B_2+1}\eta_1^* \\
\Rightarrow \eta_3 &= (-1)^{(A_1-B_1)+(A_2-B_2)+1}\eta_1 \\
\Rightarrow \eta_3 &\neq (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1
\end{aligned}$$

and

$$\begin{aligned}
&l_3'^* - l_1'^* < (A_3^* - B_3^*)/2 - (A_1 - B_1) + l_1'^* \\
\Rightarrow l_3^* - l_1^* &< (A_3 - B_3)/2 + (B_3 - B_1) - (A_1 - B_1) + l_1^* \\
\Rightarrow l_3 + (B_3 - B_1) - l_1 &< (A_3 - B_3)/2 + (B_3 - B_1) - (A_1 - B_1) + l_1 \\
\Rightarrow l_3 - l_1 &< (A_3 - B_3)/2 - (A_1 - B_1) + l_1
\end{aligned}$$

We also have

$$\begin{aligned}
(-1)^{A_3-B_3}\eta_3''^* &\neq \eta_1''^* = (-1)^{A_3-B_3}\eta_1'^* \\
\Rightarrow \eta_3''^* &= -\eta_1'^* \\
\Rightarrow \eta_3''^* &= -(-1)^{A_2-B_2+1}\eta_1^* \\
\Rightarrow \eta_3''^* &= (-1)^{A_2-B_2}\eta_1
\end{aligned}$$

and

$$l_3''^* = l_3'^* + (A_1 - B_1 - 2l_1'^*) + 1$$

$$\begin{aligned}
&= l_3^* + (A_1 - B_1 - 2l_1^*) + 1 \\
&= l_3 + (B_3 - B_1) + (A_1 - B_1 - 2l_1) + 1 \\
&= l_3 - 2l_1 + A_1 + B_3 - 2B_1 + 1
\end{aligned}$$

So we will get the following conditions from (8.3).

– If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, then

$$0 \leq (l_3 - 2l_1 + A_1 + B_3 - 2B_1 + 1) + B_1 - l_2 \leq (A_3 + B_3 - 0) - (A_2 - B_2)$$

which implies

$$-(A_1 - B_1) - B_3 - 1 \leq l_3 - l_2 - 2l_1 \leq A_3 - (A_1 - B_1) - (A_2 - B_2) - 1$$

– If $\eta_2 = (-1)^{A_1-B_1}\eta_1$, then

$$(l_3 - 2l_1 + A_1 + B_3 - 2B_1 + 1) + B_1 + l_2 > A_2 - B_2$$

which implies

$$l_3 + l_2 - 2l_1 > (A_2 - B_2) - (A_1 - B_1) - B_3 - 1$$

• If $\eta_3^* = (-1)^{A_1-B_1}\eta_1^*$ and

$$l_3^* - l_1^* \geq (A_3^* - B_3^*)/2 - (A_1 - B_1) + l_1^*,$$

then $\eta_1''^* = (-1)^{A_3-B_3}\eta_3''^*$ and

$$\begin{cases} l_1^* = l_1''^* \\ (l_3''^* - l_1''^*) + (l_3^* - l_1^*) = (A_3^* - B_3^*) - (A_1 - B_1) \\ \eta_1''^* = (-1)^{A_3-B_3}\eta_1'^* \end{cases}$$

It follows

$$\begin{aligned}
&\eta_3'^* = (-1)^{A_1-B_1}\eta_1'^* \\
&\Rightarrow \eta_3^* = (-1)^{A_1-B_1}(-1)^{A_2-B_2+1}\eta_1^* \\
&\Rightarrow \eta_3 = (-1)^{(A_1-B_1)+(A_2-B_2)+1}\eta_1 \\
&\Rightarrow \eta_3 \neq (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1
\end{aligned}$$

and

$$\begin{aligned}
&l_3^* - l_1^* \geq (A_3^* - B_3^*)/2 - (A_1 - B_1) + l_1^* \\
&\Rightarrow l_3^* - l_1^* \geq (A_3 - B_3)/2 + (B_3 - B_1) - (A_1 - B_1) + l_1^* \\
&\Rightarrow l_3 + (B_3 - B_1) - l_1 \geq (A_3 - B_3)/2 + (B_3 - B_1) - (A_1 - B_1) + l_1 \\
&\Rightarrow l_3 - l_1 \geq (A_3 - B_3)/2 - (A_1 - B_1) + l_1
\end{aligned}$$

We also have

$$\begin{aligned}
&(-1)^{A_3-B_3}\eta_3''^* = \eta_1''^* = (-1)^{A_3-B_3}\eta_1'^* \\
&\Rightarrow \eta_3''^* = \eta_1'^* \\
&\Rightarrow \eta_3''^* = (-1)^{A_2-B_2+1}\eta_1^* \\
&\Rightarrow \eta_3''^* = (-1)^{A_2-B_2+1}\eta_1
\end{aligned}$$

and

$$\begin{aligned}
l_3''^* &= l_1''^* - (l_3^* - l_1^*) + (A_3^* - B_3^*) - (A_1 - B_1) \\
&= 2l_1^* - l_3^* + (A_3 - B_3) + 2(B_3 - B_1) - (A_1 - B_1) \\
&= 2l_1^* - l_3^* + (A_3 - B_3) + 2(B_3 - B_1) - (A_1 - B_1)
\end{aligned}$$

$$\begin{aligned}
&= 2l_1 - l_3 - (B_3 - B_1) + (A_3 - B_3) + 2(B_3 - B_1) - (A_1 - B_1) \\
&= 2l_1 - l_3 + (A_3 - B_3) + (B_3 - B_1) - (A_1 - B_1) \\
&= 2l_1 - l_3 + A_3 - A_1
\end{aligned}$$

So we will get the following conditions from (8.3).

– If $\eta_2 = (-1)^{A_1-B_1}\eta_1$, then

$$0 \leq (2l_1 - l_3 + A_3 - A_1) + B_1 - l_2 \leq (A_3 + B_3 - 0) - (A_2 - B_2)$$

which implies

$$(A_1 - B_1) - A_3 \leq -l_3 - l_2 + 2l_1 \leq (A_1 - B_1) - (A_2 - B_2) + B_3$$

– If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, then

$$(2l_1 - l_3 + A_3 - A_1) + B_1 + l_2 > A_2 - B_2$$

which implies

$$-l_3 + l_2 + 2l_1 > (A_1 - B_1) + (A_2 - B_2) - A_3$$

To summarize, we obtain the following conditions for $\pi_{>\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$.

(1) If $\eta_3 = (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1$ and $\eta_2 = (-1)^{A_1-B_1}\eta_1$, then

$$\begin{cases} l_3 + l_1 > A_1 - B_3 \\ -B_2 \leq l_2 - l_1 \leq A_2 - (A_1 - B_1) \\ (A_1 - B_1) - B_3 + 1 \leq l_3 - l_2 + 2l_1 \leq A_3 + (A_1 - B_1) - (A_2 - B_2) + 1 \end{cases}$$

(2) If $\eta_3 = (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1$ and $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, then

$$\begin{cases} l_3 + l_1 > A_1 - B_3 \\ l_1 + l_2 > (A_1 - B_1) - B_2 \\ l_3 + l_2 + 2l_1 > (A_1 - B_1) + (A_2 - B_2) - B_3 + 1 \end{cases}$$

(3) If $\eta_3 \neq (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1$, $\eta_2 = (-1)^{A_1-B_1}\eta_1$, and

$$l_3 - l_1 < (A_3 - B_3)/2 - (A_1 - B_1) + l_1$$

then

$$\begin{cases} -(B_3 - B_1) \leq l_3 - l_1 \leq A_3 - A_1 \\ -B_2 \leq l_2 - l_1 \leq A_2 - (A_1 - B_1) \\ l_3 + l_2 - 2l_1 > (A_2 - B_2) - (A_1 - B_1) - B_3 - 1 \end{cases}$$

(4) If $\eta_3 \neq (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1$, $\eta_2 = (-1)^{A_1-B_1}\eta_1$, and

$$l_3 - l_1 \geq (A_3 - B_3)/2 - (A_1 - B_1) + l_1$$

then

$$\begin{cases} -(B_3 - B_1) \leq l_3 - l_1 \leq A_3 - A_1 \\ -B_2 \leq l_2 - l_1 \leq A_2 - (A_1 - B_1) \\ (A_1 - B_1) - A_3 \leq -l_3 - l_2 + 2l_1 \leq (A_1 - B_1) - (A_2 - B_2) + B_3 \end{cases}$$

(5) If $\eta_3 \neq (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1$, $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, and

$$l_3 - l_1 < (A_3 - B_3)/2 - (A_1 - B_1) + l_1$$

then

$$\begin{cases} -(B_3 - B_1) \leq l_3 - l_1 \leq A_3 - A_1 \\ l_1 + l_2 > (A_1 - B_1) - B_2 \\ -(A_1 - B_1) - B_3 - 1 \leq l_3 - l_2 - 2l_1 \leq A_3 - (A_1 - B_1) - (A_2 - B_2) - 1 \end{cases}$$

(6) If $\eta_3 \neq (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1$, $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, and

$$l_3 - l_1 \geq (A_3 - B_3)/2 - (A_1 - B_1) + l_1$$

then

$$\begin{cases} -(B_3 - B_1) \leq l_3 - l_1 \leq A_3 - A_1 \\ l_1 + l_2 > (A_1 - B_1) - B_2 \\ -l_3 + l_2 + 2l_1 > (A_1 - B_1) + (A_2 - B_2) - A_3 \end{cases}$$

To get the complete conditions, we still need to include

$$0 \leq l_i \leq [(A_i - B_i + 1)/2],$$

and

$$\prod_{i=1}^3 \eta_i^{A_i-B_i+1} (-1)^{[(A_i-B_i+1)/2]+l_i} = 1$$

(cf. (1.9)).

Example 8.1. Let $[A_3, B_3] = [40, 10]$, $[A_2, B_2] = [37, 7]$ and $[A_1, B_1] = [8, 4]$, i.e.,

$$\psi = \rho \otimes \nu_{51} \otimes \nu_{31} \oplus \rho \otimes \nu_{31} \otimes \nu_{45} \oplus \rho \otimes \nu_{13} \otimes \nu_5.$$

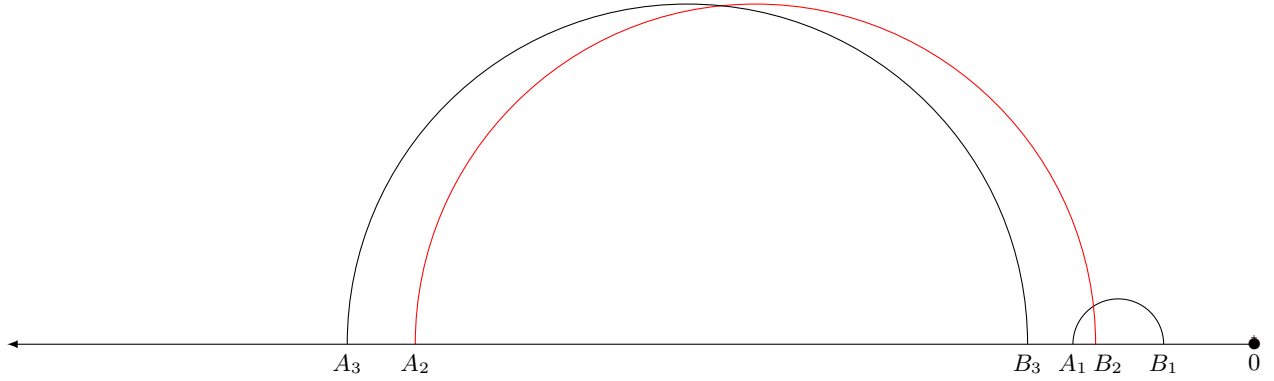


FIGURE 8. ψ

First, we have $0 \leq l_1 \leq 2, 0 \leq l_2 \leq 15, 0 \leq l_3 \leq 15$, and $(-1)^{l_1+l_2+l_3}\eta_1\eta_2\eta_3 = 1$. Note $B_3 > A_1$, so the conditions from (8.1) are always satisfied. Also note $B_2 > A_1 - B_1$, then the conditions from (8.2) are always satisfied too. Therefore, we can simplify the nonvanishing conditions as follows:

(1) If $\eta_3 = \eta_1$ and $\eta_2 = \eta_1$, then

$$-5 \leq l_3 - l_2 + 2l_1 \leq 15$$

(2) If $\eta_3 = \eta_1$ and $\eta_2 \neq \eta_1$, then

$$l_3 + l_2 + 2l_1 > 25$$

(3) If $\eta_3 \neq \eta_1, \eta_2 = \eta_1$, and

$$l_3 - l_1 < 11 + l_1$$

then

$$l_3 + l_2 - 2l_1 > 15$$

(4) If $\eta_3 \neq \eta_1, \eta_2 \neq \eta_1$, and

$$l_3 - l_1 \geq 11 + l_1$$

then

$$-36 \leq -l_3 - l_2 + 2l_1 \leq -16$$

(5) If $\eta_3 \neq \eta_1$, $\eta_2 \neq \eta_1$, and

$$l_3 - l_1 < 11 + l_1$$

then

$$-15 \leq l_3 - l_2 - 2l_1 \leq 5$$

(6) If $\eta_3 \neq \eta_1$, $\eta_2 \neq \eta_1$, and

$$l_3 - l_1 \geq 11 + l_1$$

then

$$-l_3 + l_2 + 2l_1 > -6$$

To find the size of $\Pi_\psi^{\Sigma_0}$ is equivalent to counting integral points in certain polytopes for each of the above six cases. By running a simple computer program, we can get $|\Pi_\psi^{\Sigma_0}| = 1651$.

APPENDIX A. PROOF OF LEMMA 4.1

In this appendix, we want to give the proof of Lemma 4.1. We will mainly follow the idea of Mœglin in her original proof (cf. [Mœg06a], Lemma 3.4). However, we think there is a “minor mistake” in the original proof, so we will also fix it here. Let us first recall the statement of Lemma 4.1.

Lemma A.1 (Mœglin). *In the basic case, suppose*

$$[A_2, B_2] = [A_1, B_1],$$

then $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if

$$\begin{cases} \eta_2 = (-1)^{A_1 - B_1} \eta_1 & \Rightarrow l_1 = l_2, \\ \eta_2 \neq (-1)^{A_1 - B_1} \eta_1 & \Rightarrow l_1 = l_2 = (A_1 - B_1 + 1)/2. \end{cases}$$

Proof. The sufficiency of this nonvanishing condition will follow from the proof of Proposition 4.2, so it suffices for us to prove its necessity here. We will prove this by induction on $A_1 - B_1$. In particular, we can assume Proposition 4.2 for $[A'_2, B'_2] \neq [A'_1, B'_1]$ such that $A'_i - B'_i \leq A_1 - B_1$ ($i = 1, 2$). Note when $A_1 - B_1 = 0$, this lemma is clear (cf. Theorem 1.1). Let us define ψ_{\gg} by shifting $(\rho, A_2, B_2, \zeta_2)$ to $(\rho, A_2 + T, B_2 + T, \zeta_2)$, such that ψ_{\gg} has discrete diagonal restriction and it admits the same order $>_\psi$. Suppose $\pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, we first want to show

$$(A.1) \quad \begin{cases} \eta_2 = (-1)^{A_1 - B_1} \eta_1 & \Rightarrow |l_1 - l_2| \leq 1, \\ \eta_2 \neq (-1)^{A_1 - B_1} \eta_1 & \Rightarrow l_1 + l_2 + 1 > A_1 - B_1. \end{cases}$$

Let us consider the following situations.

- (1) If $l_1 = l_2 = 0$, it is clear that one must have $\eta_2 = (-1)^{A_1 - B_1} \eta_1$.
- (2) If $l_1 \neq 0$, then

$$\begin{aligned} \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow < \zeta B_1, \dots - \zeta A_1 > \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; \\ (\rho, A_2 + T, B_2 + T, l_2, \eta_2, \zeta), (\rho, A_1 - 1, B_1 + 1, l_1 - 1, \eta_1, \zeta) \Big). \end{aligned}$$

Note

$$\text{Jac}_{(\rho, A_2 + T, B_2 + T, \zeta) \mapsto (\rho, A_2, B_2, \zeta)} \pi_{M, >_\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) = \pi_{M, >_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0.$$

So after applying $\text{Jac}_{(\rho, A_2 + T, B_2 + T, \zeta) \mapsto (\rho, A_2, B_2, \zeta)}$ to the full induced representation above, we have

$$< \zeta B_1, \dots - \zeta A_1 > \rtimes \pi_{M, >_\psi}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; (\rho, A_2, B_2, l_2, \eta_2, \zeta), (\rho, A_1 - 1, B_1 + 1, l_1 - 1, \eta_1, \zeta) \Big),$$

which is again nonzero. In particular,

$$\pi_{M, >_\psi}^{\Sigma_0}(\psi_{-}, \underline{l}_{-}, \underline{\eta}_{-}; (\rho, A_2, B_2, l_2, \eta_2, \zeta), (\rho, A_1 - 1, B_1 + 1, l_1 - 1, \eta_1, \zeta) \Big) \neq 0.$$

Here we will only need the weak fact that

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 + 1, B_2 + 1, l_2, \eta_2, \zeta), (\rho, A_1 - 1, B_1 + 1, l_1 - 1, \eta_1, \zeta)) \neq 0.$$

By our induction assumption, we can conclude

$$\begin{cases} \eta_2 = (-1)^{(A_1-1)-(B_1+1)}\eta_1 & \Rightarrow (A_2 + 1) - l_2 \geq (A_1 - 1) - (l_1 - 1), \\ & (B_2 + 1) + l_2 \geq (B_1 + 1) + (l_1 - 1); \\ \eta_2 \neq (-1)^{(A_1-1)-(B_1+1)}\eta_1 & \Rightarrow (B_2 + 1) + l_2 > (A_1 - 1) - (l_1 - 1). \end{cases}$$

In the first case, we get $\eta_2 = (-1)^{A_1-B_1}\eta_1$ and $-1 \leq l_2 - l_1 \leq 1$. In the second case, we have $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$ and $l_1 + l_2 + 1 > A_1 - B_2 = A_1 - B_1$.

(3) If $l_2 \neq 0$, then

$$\begin{aligned} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow < \zeta(B_2 + T), \dots - \zeta(A_2 + T) > \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; \\ (\rho, A_2 + T - 1, B_2 + T + 1, l_2 - 1, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)). \end{aligned}$$

Note

$$\text{Jac}_{(\rho, A_2+T, B_2+T, \zeta) \mapsto (\rho, A_2, B_2, \zeta)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) = \pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0.$$

So after applying $\text{Jac}_{(\rho, A_2+T, B_2+T, \zeta) \mapsto (\rho, A_2, B_2, \zeta)}$ to the full induced representation above, we have

$$< \zeta B_2, \dots - \zeta A_2 > \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - 1, B_2 + 1, l_2 - 1, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)),$$

which is again nonzero. In particular,

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - 1, B_2 + 1, l_2 - 1, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \neq 0.$$

Here we will only need the weak fact that

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2, B_2 + 2, l_2 - 1, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \neq 0.$$

By our induction assumption, we can conclude

$$\begin{cases} \eta_2 = (-1)^{A_1-B_1}\eta_1 & \Rightarrow A_2 - (l_2 - 1) \geq A_1 - l_1, \\ & (B_2 + 2) + (l_2 - 1) \geq B_1 + l_1; \\ \eta_2 \neq (-1)^{A_1-B_1}\eta_1 & \Rightarrow (B_2 + 2) + (l_2 - 1) > A_1 - l_1. \end{cases}$$

In the first case, we get $\eta_2 = (-1)^{A_1-B_1}\eta_1$ and $-1 \leq l_2 - l_1 \leq 1$. In the second case, we have $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$ and $l_1 + l_2 + 1 > A_1 - B_2 = A_1 - B_1$.

Now we will assume (A.1). If $\eta_2 = (-1)^{A_1-B_1}\eta_1$, we still need to eliminate the case $|l_1 - l_2| = 1$. If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, we need to eliminate the following cases:

- (1) $|l_1 - l_2| = 1$, $\max\{l_1, l_2\} = (A_1 - B_1 + 1)/2$.
- (2) $l_1 = l_2 = (A_1 - B_1)/2$.

To simplify the notations, we let $A = A_1 = A_2$ and $B = B_1 = B_2$.

A.1. Case: $l_1 = l_2 + 1$. Let us denote l_2 by l . Since $A - l_1 + 1 > B + l_1 - 1$, then $A - l > B + l$.

- (1) $A - l > B + l + 1$.

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1 + l_1 - 1) & \cdots & -\zeta(A_1 - l_1 + 1) \end{pmatrix}$$

$$\begin{aligned}
& \times \begin{pmatrix} \zeta(B_2 + T) & \cdots & -\zeta(A_2 + T) \\ \vdots & & \vdots \\ \zeta(B_2 + l_2 - 1 + T) & \cdots & -\zeta(A_2 - l_2 + 1 + T) \end{pmatrix} \\
& \times \begin{pmatrix} \zeta(B_2 + l_2 + T) & \cdots & -\zeta(A_1 - l_1) \\ \vdots & & \vdots \\ \zeta(A_2 - l_2 - 2 + T) & \cdots & -\zeta(B_1 + l_1) \end{pmatrix} \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - l_2 + T, A_2 - l_2 - 1 + T, 0, (-1)^{A_2 - B_2 - 1} \eta_2, \zeta)) \\
& \hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B + l) & \cdots & -\zeta(A - l) \end{pmatrix}}_{*-1} \times \underbrace{\begin{pmatrix} \zeta(B + T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B + l - 1 + T) & \cdots & -\zeta(A - l + 1) \end{pmatrix}}_{*-2} \\
& \times \underbrace{\begin{pmatrix} -\zeta(A + 1) & \cdots & -\zeta(A + T) \\ \vdots & & \vdots \\ -\zeta(A - l + 2) & \cdots & -\zeta(A - l + 1 + T) \end{pmatrix}}_I \\
& \times \underbrace{\begin{pmatrix} \zeta(B + l + T) & \cdots & -\zeta(A - l - 1) \\ \vdots & & \vdots \\ \zeta(A - l - 2 + T) & \cdots & -\zeta(B + l + 1) \end{pmatrix}}_{II} \\
& \times \underbrace{\begin{pmatrix} \zeta(A - l - 1 + T) & \cdots & \zeta(A - l) \\ \zeta(A - l + T) & \cdots & \zeta(A - l + 1) \end{pmatrix}}_{III} \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A - l, A - l - 1, 0, (-1)^{A_2 - B_2 - 1} \eta_2, \zeta))
\end{aligned}$$

We can interchange (I) with (II) and (III). Note

$$(I) \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A - l, A - l - 1, 0, (-1)^{A_2 - B_2 - 1} \eta_2, \zeta))$$

is irreducible (see Proposition 3.6), so we can also take dual of (I) (see Corollary 3.7). Moreover, $(* - 1)$ and $(* - 2)$ are interchangeable. Therefore,

$$\begin{aligned}
\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) & \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B + T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B + l - 1 + T) & \cdots & -\zeta(A - l + 1) \end{pmatrix}}_{*-2} \times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B + l) & \cdots & -\zeta(A - l) \end{pmatrix}}_{*-1} \\
& \times \underbrace{\begin{pmatrix} \zeta(B + l + T) & \cdots & -\zeta(A - l - 1) \\ \vdots & & \vdots \\ \zeta(A - l - 2 + T) & \cdots & -\zeta(B + l + 1) \end{pmatrix}}_{II} \\
& \times \underbrace{\begin{pmatrix} \zeta(A - l - 1 + T) & \cdots & \zeta(A - l) \\ \zeta(A - l + T) & \cdots & \zeta(A - l + 1) \end{pmatrix}}_{III}
\end{aligned}$$

$$\times \underbrace{\begin{pmatrix} \zeta(A-l+1+T) & \cdots & \zeta(A-l+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+1) \end{pmatrix}}_{(I)^\vee} \\ \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta)).$$

We can “combine” (II) with (III), for otherwise $\text{Jac}_{\zeta(A-l+1+T)} \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \neq 0$, which is impossible. Here we have used the fact $A-l > B+l+1$, in order to switch $\rho ||^{\zeta(A-l+1+T)}$ with $(*-2)$. For the same kind of reason, we can “combine” (III) with $(I)^\vee$. Consequently,

$$\pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-2} \times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l) \end{pmatrix}}_{*-1} \\ \times \underbrace{\begin{pmatrix} \zeta(B+l+T) & \cdots & \zeta(B+l+1) & \cdots & -\zeta(A-l-1) \\ \vdots & & \vdots & & \vdots \\ \zeta(A-l-2+T) & \cdots & \zeta(A-l-1) & \cdots & -\zeta(B+l+1) \\ \zeta(A-l-1+T) & \cdots & \zeta(A-l) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+1) \end{pmatrix}}_{IV} \\ \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta)).$$

We can further “combine” $(*-1)$ with (IV), for otherwise $\text{Jac}_{\zeta(A-l)} \pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \neq 0$, which is again impossible. Here we have used the fact that

$$(A.2) \quad \rho ||^{-\zeta(A-l)} \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta))$$

is irreducible (we will prove it in end of this case), and $A-l > B+l+1$. As a result,

$$\pi_{M,>\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-2} \\ \times \underbrace{\begin{pmatrix} & \zeta B & \cdots & -\zeta A \\ & \vdots & & \vdots \\ & \zeta(B+l) & \cdots & -\zeta(A-l) \\ \zeta(B+l+T) & \cdots & \zeta(B+l+1) & \cdots & -\zeta(A-l-1) \\ \vdots & & \vdots & & \vdots \\ \zeta(A-l-2+T) & \cdots & \zeta(A-l-1) & \cdots & -\zeta(B+l+1) \\ \zeta(A-l-1+T) & \cdots & \zeta(A-l) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+1) \end{pmatrix}}_{(*-1)+IV} \\ \rtimes \pi_{M,>\psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta)).$$

Hence

$$\begin{aligned}
& \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 + 1, B_2 + 1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta) \right) \\
& \hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l+1) \end{pmatrix} \\
& \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l-1) & \cdots & -\zeta(B+l+1) \\ \vdots & & \vdots \\ \zeta(A+1) \end{pmatrix} \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right).
\end{aligned}$$

If we apply $\text{Jac}_{(\rho, A+1, B+1, \zeta) \mapsto (\rho, A, B, \zeta)}$ to the full induced representation above, we should get zero. This means $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = 0$.

To complete the discussion of this case, we still need to show (A.2) is irreducible. We will use the criterion of Lemma 3.5. Since

$$\text{Jac}_{-\zeta(A-l)}(\text{A.2}) = \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right),$$

we see (A.2) has a unique subrepresentation σ , and σ is multiplicity free in $s.s.(\text{A.2})$. Since

$$\text{Jac}_{\zeta(A-l)}(\text{A.2}) = \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right),$$

it suffices to show $\text{Jac}_{\zeta(A-l)}\sigma \neq 0$. Note $A-l-2 > B+l-1$, so

$$\begin{aligned}
\sigma & \hookrightarrow \rho ||^{-\zeta(A-l)} \times \rho ||^{\zeta(A-l-1)} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l, 0, (-1)^{A_2-B_2} \eta_2, \zeta), \right. \\
& \quad \left. (\rho, A-l-2, A-l-2, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right) \\
& \cong \rho ||^{\zeta(A-l-1)} \times \rho ||^{-\zeta(A-l)} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l, 0, (-1)^{A_2-B_2} \eta_2, \zeta), \right. \\
& \quad \left. (\rho, A-l-2, A-l-2, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right) \\
& \cong \rho ||^{\zeta(A-l-1)} \times \rho ||^{\zeta(A-l)} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l, 0, (-1)^{A_2-B_2} \eta_2, \zeta), \right. \\
& \quad \left. (\rho, A-l-2, A-l-2, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right).
\end{aligned}$$

The last isomorphism does not follow from Lemma 3.4 exactly, but one can prove it using the same argument there together with ([Møeg06b], Proposition 2.7). If

$$\begin{aligned}
\sigma & \hookrightarrow \langle \zeta(A-l), \zeta(A-l-1) \rangle \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l, 0, (-1)^{A_2-B_2} \eta_2, \zeta), \right. \\
& \quad \left. (\rho, A-l-2, A-l-2, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right),
\end{aligned}$$

then it is clear that $\text{Jac}_{\zeta(A-l)}\sigma \neq 0$. Otherwise, we have

$$\begin{aligned}
\sigma & \hookrightarrow \langle \zeta(A-l-1), \zeta(A-l) \rangle \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; \right. \\
& \quad \left. (\rho, A-l, A-l, 0, (-1)^{A_2-B_2} \eta_2, \zeta), (\rho, A-l-2, A-l-2, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right) \\
& \hookrightarrow \langle \zeta(A-l-1), \zeta(A-l) \rangle \times \rho ||^{\zeta(A-l)} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; \right. \\
& \quad \left. (\rho, A-l-1, A-l-1, 0, (-1)^{A_2-B_2} \eta_2, \zeta), (\rho, A-l-2, A-l-2, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right)
\end{aligned}$$

$$\cong \rho ||^{\zeta(A-l)} \times \langle \zeta(A-l-1), \zeta(A-l) \rangle \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; \right. \\ \left. (\rho, A-l-1, A-l-2, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right).$$

So we again have $\text{Jac}_{\zeta(A-l)} \sigma \neq 0$. This finishes the proof.

(2) $A-l = B+l+1$.

Following the previous discussion, we find (II) is “missing”, but we can still “combine” (III) and (I)[∨].

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-2} \times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l) \end{pmatrix}}_{*-1} \\ \times \underbrace{\begin{pmatrix} \zeta(A-l-1+T) & \cdots & \zeta(A-l) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+1) \end{pmatrix}}_{IV} \\ \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right).$$

Hence

$$\pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta) \right) \\ \hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l) \end{pmatrix} \\ \times \begin{pmatrix} \zeta(A-l) \\ \vdots \\ \zeta(A+1) \end{pmatrix} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right).$$

We claim the induced representation above has a unique irreducible subrepresentation. It is clear that for any irreducible subrepresentation σ , one has

$$\sigma \hookrightarrow \langle \zeta B, \cdots, -\zeta A \rangle \times \langle \zeta(B+1), \cdots, -\zeta A \rangle \times \cdots \times \langle \zeta(B+l-1), \cdots, -\zeta(A-l+1) \rangle \\ \times \langle \zeta(B+l), \cdots, -\zeta(A-l+1) \rangle \times \langle \zeta(B+l), \cdots, -\zeta(A-l) \rangle \\ \times \begin{pmatrix} \zeta(A-l) \\ \vdots \\ \zeta(A+1) \end{pmatrix} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right).$$

So the corresponding Jacquet module of σ under

$$\text{Jac}_X := \text{Jac}_{\zeta(A-l), \dots, \zeta(A+1)} \circ \text{Jac}_{\zeta(B+l), \dots, -\zeta(A-l)} \circ \text{Jac}_{\zeta(B+l), \dots, -\zeta(A-l+1)} \circ \\ \text{Jac}_{\zeta(B+l-1), \dots, -\zeta(A-l+1)} \circ \cdots \circ \text{Jac}_{\zeta(B+1), \dots, -\zeta A} \circ \text{Jac}_{\zeta B, \dots, -\zeta A}$$

contains the irreducible representation $\pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right)$.

On the other hand, we can also apply Jac_X to the full induced representation

$$\begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l) \end{pmatrix}$$

$$\times \begin{pmatrix} \zeta(A-l) \\ \vdots \\ \zeta(A+1) \end{pmatrix} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right),$$

and we get $\pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right)$. This proves our claim. As a result,

$$\begin{aligned} & \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta) \right) \\ & \hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l) \\ \vdots & & \vdots \\ \zeta(A+1) \end{pmatrix} \\ & \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1} \eta_2, \zeta) \right). \end{aligned}$$

Therefore, if we apply $\text{Jac}_{(\rho, A+1, B+1, \zeta) \mapsto (\rho, A, B, \zeta)}$ to the full induced representation above, we should get zero. This means $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = 0$.

A.2. Case: $l_2 = l_1 + 1$. Let us denote l_1 by l . Since $A - l_2 + 1 > B + l_2 - 1$, then $A - l > B + l$.

(1) $l \neq 0$ and $A - l > B + l + 1$.

$$\begin{aligned} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) & \hookrightarrow \begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1 + l_1 - 1) & \cdots & -\zeta(A_1 - l_1 + 1) \end{pmatrix} \\ & \times \begin{pmatrix} \zeta(B_2 + T) & \cdots & -\zeta(A_2 + T) \\ \vdots & & \vdots \\ \zeta(B_2 + l_2 - 1 + T) & \cdots & -\zeta(A_2 - l_2 + 1 + T) \end{pmatrix} \\ & \times \begin{pmatrix} \zeta(B_2 + l_2 + T) & \cdots & -\zeta(A_1 - l_1) \\ \vdots & & \vdots \\ \zeta(A_2 - l_2 + T) & \cdots & -\zeta(B_1 + l_1 + 2) \end{pmatrix} \\ & \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B_1 + l_1 + 1, B_1 + l_1, 0, \eta_1, \zeta) \right) \\ & \hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1} \times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta(A+1) \\ \vdots & & \vdots \\ \zeta(B+l+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_I \\ & \times \underbrace{\begin{pmatrix} -\zeta(A+2) & \cdots & -\zeta(A+T) \\ \vdots & & \vdots \\ -\zeta(A-l+2) & \cdots & -\zeta(A-l+T) \end{pmatrix}}_{II} \end{aligned}$$

$$\begin{aligned}
& \times \underbrace{\begin{pmatrix} \zeta(B+l+1+T) & \cdots & -\zeta(A-l) \\ \vdots & & \vdots \\ \zeta(A-l-1+T) & \cdots & -\zeta(B+l+2) \end{pmatrix}}_{III} \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta)).
\end{aligned}$$

We first interchange (II) and (III), then take dual of (II) (see Corollary 3.7).

$$\begin{aligned}
\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) & \hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1} \times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta(A+1) \\ \vdots & & \vdots \\ \zeta(B+l+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_I \\
& \times \underbrace{\begin{pmatrix} \zeta(B+l+1+T) & \cdots & -\zeta(A-l) \\ \vdots & & \vdots \\ \zeta(A-l-1+T) & \cdots & -\zeta(B+l+2) \end{pmatrix}}_{III} \\
& \times \underbrace{\begin{pmatrix} \zeta(A-l+T) & \cdots & \zeta(A-l+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix}}_{(II)^\vee} \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta)).
\end{aligned}$$

Since $\text{Jac}_{\zeta(B+l+1+T)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) = 0$, we can “combine” (I) and (III). For the same kind of reason, we can further “combine” them with (II)[∨]. So

$$\begin{aligned}
\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) & \hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1} \\
& \times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta(A+1) \\ \vdots & & \vdots & & \vdots \\ \zeta(A-l-1+T) & \cdots & \zeta(A-l+1) & \cdots & -\zeta(B+l+2) \\ \zeta(A-l+T) & \cdots & \zeta(A-l+2) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{*-2} \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta)) \\
& \hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1}
\end{aligned}$$

$$\begin{aligned}
& \times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta A \\ \vdots & & \vdots & & \vdots \\ \zeta(A-l-1+T) & \cdots & \zeta(A-l+1) & \cdots & -\zeta(B+l+1) \\ \zeta(A-l+T) & \cdots & \zeta(A-l+2) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)_-} \\
& \times \underbrace{\begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+2) \end{pmatrix}}_{IV} \times \underbrace{\begin{pmatrix} \zeta(B+l) \\ \zeta(B+l+1) \end{pmatrix}}_V \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l-1, 0, \eta_1, \zeta))
\end{aligned}$$

We interchange $(*-1)$ and $(*-2)_-$, also (IV) and (V) . Then we take dual of (IV) .

$$\begin{aligned}
\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) & \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta A \\ \vdots & & \vdots & & \vdots \\ \zeta(A-l-1+T) & \cdots & \zeta(A-l+1) & \cdots & -\zeta(B+l+1) \\ \zeta(A-l+T) & \cdots & \zeta(A-l+2) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)_-} \\
& \times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1} \\
& \times \underbrace{\begin{pmatrix} \zeta(B+l) \\ \zeta(B+l+1) \end{pmatrix}}_V \times \underbrace{\begin{pmatrix} \zeta(B+l+2) \\ \vdots \\ \zeta(A+1) \end{pmatrix}}_{(IV)^\vee} \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l-1, 0, \eta_1, \zeta)).
\end{aligned}$$

We can “combine” $(*-1)$ and (V) for $\text{Jac}_{\zeta(B+l)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) = 0$. We can also “combine” (V) and $(IV)^\vee$ for $\text{Jac}_{\zeta(B+l+2)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) = 0$. Here we have used the fact that $A-l > B+l+1$. So

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta A \\ \vdots & & \vdots & & \vdots \\ \zeta(A-l-1+T) & \cdots & \zeta(A-l+1) & \cdots & -\zeta(B+l+1) \\ \zeta(A-l+T) & \cdots & \zeta(A-l+2) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)_-}$$

$$\begin{aligned}
& \times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \\ \zeta(B+l) & & \\ \vdots & & \\ \zeta(A+1) \end{pmatrix}}_{(*-1)_+} \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l-1, 0, \eta_1, \zeta)).
\end{aligned}$$

Then

$$\begin{aligned}
& \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \\
& \hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l) & \cdots & -\zeta(B+l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \\ \zeta(B+l) & & \\ \vdots & & \\ \zeta(A+1) \end{pmatrix} \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l-1, 0, \eta_1, \zeta)).
\end{aligned}$$

Therefore, if we apply $\text{Jac}_{(\rho, A+1, B+1, \zeta) \mapsto (\rho, A, B, \zeta)}$ to the full induced representation above, we should get zero. This means $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = 0$.

(2) $l \neq 0$ and $A-l = B+l+1$.

It follows from the previous discussion that

$$\begin{aligned}
& \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1} \\
& \times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta(A+1) \\ \vdots & & \vdots & & \vdots \\ \zeta(A-l-1+T) & \cdots & \zeta(A-l+1) & \cdots & -\zeta(B+l+2) \\ \zeta(A-l+T) & \cdots & \zeta(A-l+2) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix}}_{*-2} \\
& \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta))
\end{aligned}$$

Since $(*-1)$ and $(*-2)$ are interchangeable, then we have

$$\begin{aligned}
& \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \\
& \hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta(A+1) \\ \vdots & & \vdots \\ \zeta(A-l) & \cdots & -\zeta(B+l+2) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}
\end{aligned}$$

$$\rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta) \right).$$

It follows

$$\begin{aligned} & \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta) \right) \\ & \hookrightarrow \langle \zeta B, \dots - \zeta A \rangle \times \langle \zeta(B+1), \dots, -\zeta(A+1) \rangle \times \dots \times \langle \zeta(B+l-1), \dots, -\zeta(A-l+1) \rangle \\ & \times \langle \zeta(A-l-1), \dots, -\zeta(B+l+3) \rangle \times \langle \zeta(A-l), \dots, -\zeta(B+l+2) \rangle \\ & \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta) \right) \end{aligned}$$

Therefore

$$(A.3) \quad \text{Jac}_X \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta) \right) \neq 0,$$

where

$$\begin{aligned} \text{Jac}_X &:= \text{Jac}_{\zeta(A-l), \dots, -\zeta(B+l+2)} \circ \text{Jac}_{\zeta(A-l-1), \dots, -\zeta(B+l+3)} \circ \\ & \text{Jac}_{\zeta(B+l-1), \dots, -\zeta(A-l+1)} \circ \dots \circ \text{Jac}_{\zeta(B+1), \dots, -\zeta(A+1)} \circ \text{Jac}_{\zeta B, \dots, -\zeta A}. \end{aligned}$$

On the other hand, we can rewrite

$$\begin{aligned} & \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta) \right) \\ & \hookrightarrow \begin{pmatrix} \zeta(B+1) & \dots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l) & \dots & -\zeta(B+l+1) \end{pmatrix} \times \underbrace{\begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+2) \end{pmatrix}}_{IV} \\ & \times \begin{pmatrix} \zeta B & \dots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \dots & -\zeta(A-l+1) \end{pmatrix} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta) \right) \\ & \cong \begin{pmatrix} \zeta(B+1) & \dots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l) & \dots & -\zeta(B+l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \dots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \dots & -\zeta(A-l+1) \end{pmatrix} \\ & \times \underbrace{\begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+2) \end{pmatrix}}_{IV} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta) \right) \end{aligned}$$

So there exists an irreducible constituent σ of

$$(IV) \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta) \right),$$

such that

$$\begin{aligned} & \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta) \right) \\ & \hookrightarrow \begin{pmatrix} \zeta(B+1) & \dots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l) & \dots & -\zeta(B+l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \dots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \dots & -\zeta(A-l+1) \end{pmatrix} \rtimes \sigma \end{aligned}$$

We claim $\text{Jac}_{\zeta(B+l+2)}\sigma = 0$. Otherwise, we necessarily have

$$\text{Jac}_{\zeta(B+l+2)}\sigma = \begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+3) \end{pmatrix} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta) \right)$$

which is irreducible by Proposition 3.6. Then

$$\begin{aligned} \sigma &\hookrightarrow \rho ||^{\zeta(B+l+2)} \times \begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+3) \end{pmatrix} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta) \right) \\ &\cong \begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+3) \end{pmatrix} \times \rho ||^{\zeta(B+l+2)} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta) \right). \end{aligned}$$

It is necessary that

$$\begin{aligned} \sigma &\hookrightarrow \begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+3) \end{pmatrix} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+2, B+l+2, 0, -\eta_1, \zeta), \right. \\ &\quad \left. (\rho, B+l, B+l, 0, \eta_1, \zeta) \right). \end{aligned}$$

In particular

$$\text{Jac}_{-\zeta(B+l+2)} \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+2, B+l+2, 0, -\eta_1, \zeta), (\rho, B+l, B+l, 0, \eta_1, \zeta) \right) = 0.$$

Now we have

$$\begin{aligned} &\pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta) \right) \\ &\hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l) & \cdots & -\zeta(B+l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix} \\ &\times \begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+3) \end{pmatrix} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+2, B+l+2, 0, -\eta_1, \zeta), \right. \\ &\quad \left. (\rho, B+l, B+l, 0, \eta_1, \zeta) \right). \end{aligned}$$

If we apply

$$\begin{aligned} \text{Jac}_{X'} &:= \text{Jac}_{\zeta(A-l-1), \dots, -\zeta(B+l+3)} \circ \text{Jac}_{\zeta(B+l-1), \dots, -\zeta(A-l+1)} \circ \cdots \circ \\ &\quad \text{Jac}_{\zeta(B+1), \dots, -\zeta(A+1)} \circ \text{Jac}_{\zeta B, \dots, -\zeta A}. \end{aligned}$$

to the full induced representation above, we will get

$$\begin{aligned} &< \zeta(A-l), \dots, -\zeta(B+l+1) > \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+2, B+l+2, 0, -\eta_1, \zeta), \right. \\ &\quad \left. (\rho, B+l, B+l, 0, \eta_1, \zeta) \right). \end{aligned}$$

Here we have used the fact that $A-l = B+l+1$. Note

$$\text{Jac}_X = \text{Jac}_{\zeta(A-l), \dots, -\zeta(B+l+2)} \circ \text{Jac}_{X'}$$

and

$$\begin{aligned} &\text{Jac}_{\zeta(A-l), \dots, -\zeta(B+l+2)} < \zeta(A-l), \dots, -\zeta(B+l+1) > \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; \right. \\ &\quad \left. (\rho, B+l+2, B+l+2, 0, -\eta_1, \zeta), (\rho, B+l, B+l, 0, \eta_1, \zeta) \right) = 0. \end{aligned}$$

This contradicts to (A.3). So we have shown our claim.

For $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ being nonzero, there necessarily exists $x \in [B+1, A+1]$ such that $\text{Jac}_{\zeta x, \dots, \zeta(A+1)} \sigma \neq 0$. By our claim, $\text{Jac}_{\zeta x} \sigma \neq 0$ implies $x = B+l$. It follows

$$\sigma \hookrightarrow \langle \zeta(B+l), \dots, \zeta(A+1) \rangle \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l-1, 0, \eta_1, \zeta)).$$

Then

$$\begin{aligned} & \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \\ & \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l) & \cdots & -\zeta(B+l+1) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1} \\ & \times \underbrace{\begin{pmatrix} \zeta(B+l) \\ \vdots \\ \zeta(A+1) \end{pmatrix}}_{VI} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l-1, 0, \eta_1, \zeta)) \end{aligned}$$

Since

$$\text{Jac}_{\zeta(B+l), \zeta(B+1)} \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) = 0,$$

we can “combine” $(*-1)$ and (VI) . So

$$\begin{aligned} & \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \\ & \hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l) & \cdots & -\zeta(B+l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \\ \zeta(B+l) & & \\ \vdots & & \\ \zeta(A+1) & & \end{pmatrix} \\ & \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l-1, 0, \eta_1, \zeta)) \end{aligned}$$

If we apply $\text{Jac}_{(\rho, A+1, B+1, \zeta) \mapsto (\rho, A, B, \zeta)}$ to the full induced representation above, we should get zero. This means $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = 0$.

(3) $l = 0$ and $A > B+1$.

From the previous discussion in (1), we have

$$\begin{aligned} & \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta(A+1) \\ \vdots & & \vdots & & \vdots \\ \zeta(A-1+T) & \cdots & \zeta(A+1) & \cdots & -\zeta(B+2) \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)} \\ & \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+1, B, 0, \eta_1, \zeta)) \end{aligned}$$

$$\hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta A \\ \vdots & & \vdots & & \vdots \\ \zeta(A-1+T) & \cdots & \zeta(A+1) & \cdots & -\zeta(B+1) \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)_-} \\ \times \underbrace{\begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+2) \end{pmatrix}}_{IV} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+1, B, 0, \eta_1, \zeta))$$

There exists an irreducible constituent σ of

$$(IV) \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+1, B, 0, \eta_1, \zeta))$$

such that

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta A \\ \vdots & & \vdots & & \vdots \\ \zeta(A-1+T) & \cdots & \zeta(A+1) & \cdots & -\zeta(B+1) \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)_-} \rtimes \sigma.$$

We claim $\text{Jac}_x \sigma = 0$ for $x \in [\zeta(B+1), \zeta(A+1)]$. This is clear when $x \neq \zeta(B+2)$. If $\text{Jac}_{\zeta(B+2)} \sigma \neq 0$, then $\text{Jac}_{\zeta(B+2)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \neq 0$, and we get a contradiction. Here we have used the fact that $A > B+1$. Note

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \\ \hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta A & \cdots & -\zeta(B+1) \end{pmatrix} \rtimes \sigma.$$

If we apply $\text{Jac}_{(\rho, A+1, B+1, \zeta) \mapsto (\rho, A, B, \zeta)}$ to the full induced representation above, we should get zero. This means $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = 0$.

(4) $l = 0$ and $A = B+1$.

We can further simplify from the previous case (3) that

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta(A+1) \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)} \\ \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+1, B, 0, \eta_1, \zeta)) \\ \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta(B+1) \\ \zeta(B+1+T) & \cdots & \zeta(B+3) & & \end{pmatrix}}_{(*-2)_-} \\ \times \rho ||^{-\zeta(B+2)} \rtimes \pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+1, B, 0, \eta_1, \zeta)).$$

So

$$\pi_{M, > \psi}^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta))$$

$$\hookrightarrow \langle \zeta(B+1), \dots, -\zeta(B+1) \rangle \times \rho \parallel^{-\zeta(B+2)} \\ \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+1, B, 0, \eta_1, \zeta) \right).$$

There exists an irreducible constituent σ of

$$\rho \parallel^{-\zeta(B+2)} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+1, B, 0, \eta_1, \zeta) \right)$$

such that

$$\pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta) \right) \\ \hookrightarrow \langle \zeta(B+1), \dots, -\zeta(B+1) \rangle \rtimes \sigma.$$

We claim $\text{Jac}_x \sigma = 0$ for $x \in [\zeta(B+1), \zeta(B+2)]$. It is clear $\text{Jac}_{\zeta(B+1)} \sigma = 0$. Suppose $\text{Jac}_{\zeta(B+2)} \sigma \neq 0$, then

$$\sigma \hookrightarrow \rho \parallel^{\zeta(B+2)} \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+1, B, 0, \eta_1, \zeta) \right)$$

as the unique irreducible subrepresentation. So

$$\sigma = \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+2, B+2, -\eta_1, \zeta), (\rho, B, B, \eta_1, \zeta) \right).$$

This implies $\text{Jac}_{-\zeta(B+2)} \sigma = 0$. In particular,

$$\text{Jac}_{\zeta(B+1), \dots, -\zeta(B+2)} \left(\langle \zeta(B+1), \dots, -\zeta(B+1) \rangle \rtimes \sigma \right) = 0.$$

On the other hand,

$$\text{Jac}_{\zeta(B+1), \dots, -\zeta(B+2)} \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta) \right) \neq 0.$$

So we get a contradiction. As a result,

$$\text{Jac}_{\zeta(B+1), \zeta(B+2)} \left(\langle \zeta(B+1), \dots, -\zeta(B+1) \rangle \rtimes \sigma \right) = 0,$$

and hence $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = 0$.

A.3. Case: $\eta_2 \neq (-1)^{A_1-B_1} \eta_1$ **and** $l_1 = l_2 = (A_1 - B_1)/2$. Let $l = l_1 = l_2 \neq 0$, then $A - l = B + l$.

$$\begin{aligned} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1 + l_1 - 1) & \cdots & -\zeta(A_1 - l_1 + 1) \end{pmatrix} \\ &\times \begin{pmatrix} \zeta(B_2 + T) & \cdots & -\zeta(A_2 + T) \\ \vdots & & \vdots \\ \zeta(B_2 + l_2 - 1 + T) & \cdots & -\zeta(A_2 - l_2 + 1 + T) \end{pmatrix} \\ &\times \langle \zeta(B_2 + l_2 + T), \dots, \zeta(B_2 + l_2 + 2) \rangle \\ &\rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B_2 + l_2 + 1, B_2 + l_2 + 1, \eta_2, \zeta), (\rho, B_1 + l_1, B_1 + l_1, \eta_1, \zeta) \right) \\ &\hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B + l - 1) & \cdots & -\zeta(A - l + 1) \end{pmatrix}}_{*-1} \times \underbrace{\begin{pmatrix} \zeta(B + T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B + l - 1 + T) & \cdots & -\zeta(A - l + 1) \end{pmatrix}}_{*-2} \\ &\times \underbrace{\begin{pmatrix} -\zeta(A + 1) & \cdots & -\zeta(A + T) \\ \vdots & & \vdots \\ -\zeta(A - l + 2) & \cdots & -\zeta(A - l + 1 + T) \end{pmatrix}}_I \end{aligned}$$

$$\begin{aligned}
& \times \underbrace{< \zeta(B+l+T), \dots \zeta(B+l+2) >}_{II} \times \underbrace{\left(\frac{\zeta(B+l)}{\zeta(B+l+1)} \right)}_{III} \\
& \times \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l, -\eta_1, \zeta), (\rho, B+l-1, B+l-1, \eta_1, \zeta) \right).
\end{aligned}$$

We interchange (I) with (II) and (III), then we take dual of (I).

$$\begin{aligned}
\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) & \hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1} \times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-2} \\
& \times \underbrace{< \zeta(B+l+T), \dots \zeta(B+l+2) >}_{II} \times \underbrace{\left(\frac{\zeta(B+l)}{\zeta(B+l+1)} \right)}_{III} \\
& \times \underbrace{\begin{pmatrix} \zeta(A-l+1+T) & \cdots & \zeta(A-l+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+1) \end{pmatrix}}_{(I)^\vee} \\
& \times \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l, -\eta_1, \zeta), (\rho, B+l-1, B+l-1, \eta_1, \zeta) \right) \\
& \hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1} \times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-2} \\
& \times \underbrace{< \zeta(B+l+T), \dots \zeta(B+l+2) >}_{II} \times \underbrace{\left(\frac{\zeta(B+l)}{\zeta(B+l+1)} \right)}_{III} \\
& \times \underbrace{\begin{pmatrix} \zeta(A-l+1+T) & \cdots & \zeta(A-l+3) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix}}_{(I)_-^\vee} \times \underbrace{\begin{pmatrix} \zeta(A-l+2) \\ \vdots \\ \zeta(A+1) \end{pmatrix}}_{IV} \\
& \times \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l, -\eta_1, \zeta), (\rho, B+l-1, B+l-1, \eta_1, \zeta) \right).
\end{aligned}$$

Note $(*-1)$ is interchangeable with $(*-2)$, (II) and $(I)_-^\vee$. And $(I)_-^\vee$ is also interchangeable with (III). So

$$\begin{aligned}
\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}, \underline{\eta}) & \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-2} \times \underbrace{< \zeta(B+l+T), \dots \zeta(B+l+2) >}_{II} \\
& \times \underbrace{\begin{pmatrix} \zeta(A-l+1+T) & \cdots & \zeta(A-l+3) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix}}_{(I)_-^\vee} \times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1}
\end{aligned}$$

$$\begin{aligned}
& \times \underbrace{\begin{pmatrix} \zeta(B+l) \\ \zeta(B+l+1) \end{pmatrix}}_{III} \times \underbrace{\begin{pmatrix} \zeta(A-l+2) \\ \vdots \\ \zeta(A+1) \end{pmatrix}}_{IV} \\
& \times \pi_{M, > \psi}^{\Sigma_0} \left(\psi_{-}, \underline{L}_{-}, \underline{\eta}_{-}; (\rho, B+l, B+l, -\eta_1, \zeta), (\rho, B+l-1, B+l-1, \eta_1, \zeta) \right).
\end{aligned}$$

Since $\text{Jac}_{\zeta(B+l+T)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{L}, \underline{\eta}) = \text{Jac}_{\zeta(A-l+1+T)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{L}, \underline{\eta}) = 0$, we can “combine” $(*-2)$, (II) and $(I)_{-}^{\vee}$.

$$\begin{aligned}
\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{L}, \underline{\eta}) & \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta A \\ \vdots & & \vdots & & \\ \zeta(B+l-1+T) & \cdots & \zeta(B+l+1) & \cdots & -\zeta(A-l+1) \\ \zeta(B+l+T) & \cdots & \zeta(B+l+2) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)_{+}} \\
& \times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1} \\
& \times \underbrace{\begin{pmatrix} \zeta(B+l) \\ \zeta(B+l+1) \end{pmatrix}}_{III} \times \underbrace{\begin{pmatrix} \zeta(A-l+2) \\ \vdots \\ \zeta(A+1) \end{pmatrix}}_{IV} \\
& \times \pi_{M, > \psi}^{\Sigma_0} \left(\psi_{-}, \underline{L}_{-}, \underline{\eta}_{-}; (\rho, B+l, B+l, -\eta_1, \zeta), (\rho, B+l-1, B+l-1, \eta_1, \zeta) \right).
\end{aligned}$$

Since $\text{Jac}_{\zeta(B+l)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{L}, \underline{\eta}) = \text{Jac}_{\zeta(A-l+2)} \pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{L}, \underline{\eta}) = 0$, we can “combine” $(*-1)$, (III) and (IV) .

$$\begin{aligned}
\pi_{M, > \psi}^{\Sigma_0}(\psi_{\gg}, \underline{L}, \underline{\eta}) & \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta A \\ \vdots & & \vdots & & \\ \zeta(B+l-1+T) & \cdots & \zeta(B+l+1) & \cdots & -\zeta(A-l+1) \\ \zeta(B+l+T) & \cdots & \zeta(B+l+2) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)_{+}} \\
& \times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \\ \zeta(B+l) & & \\ \vdots & & \\ \zeta(A+1) & & \end{pmatrix}}_{(*-1)_{+}}
\end{aligned}$$

$$\rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l, -\eta_1, \zeta), (\rho, B+l-1, B+l-1, \eta_1, \zeta) \right).$$

Then

$$\begin{aligned} & \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta) \right) \\ & \hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \\ \zeta(B+l) & & \\ \vdots & & \\ \zeta(A+1) & & \end{pmatrix} \\ & \rtimes \pi_{M, > \psi}^{\Sigma_0} \left(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l, -\eta_1, \zeta), (\rho, B+l-1, B+l-1, \eta_1, \zeta) \right). \end{aligned}$$

Therefore, if we apply $\text{Jac}_{(\rho, A+1, B+1, \zeta) \mapsto (\rho, A, B, \zeta)}$ to the full induced representation above, we should get zero. This means $\pi_{M, > \psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = 0$. □

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